University of Illinois **Fall 2025**

ECE 313: Problem Set 9: Problems and Solutions

Due: Sunday, Nov 9 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.6.3 - 3.8.2

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. **[Gaussian Approximation]**
   You play a game in which, with probability 0.55, you win \$2, and with probability 0.45, you lose \$2. You decide to play this game independently 80 times. Let $X_i \in \{-2, 2\}$ denote your earnings from the ith game, for $i = 1, \ldots, 80$. Assume $X_1, X_2, \ldots, X_{80}$ are all independent. Define
   $$X = \sum_{i=1}^{80} X_i,$$
   your total earnings (which may be negative).

   (a) Let $Z_i = (X_i + 2)/4$, for $i = 1, \ldots, 80$, and let
   $$Z = \sum_{i=1}^{80} Z_i.$$
   Interpret Z_i and determine the distribution of Z.
   Solution: Notice that Z_i is a binary indicator variable that equals 1 if the ith game was won and 0 otherwise. Therefore, each $Z_i \sim \text{Bernoulli}(p = 0.55)$, and
   $$Z \sim \text{Binomial}(n = 80,\, p = 0.55).$$

   (b) Express the event $\{X \geq 40\}$ in terms of Z, and use the Gaussian approximation with continuity correction to estimate $P\{X \geq 40\}$.
   Solution: We can relate X and Z by noting that
   $$X_i = 4Z_i - 2 \quad \Rightarrow \quad X = \sum_{i=1}^{80} (4Z_i - 2) = 4Z - 160.$$

Hence, the event $\{X \geq 40\}$ is equivalent to

$$4Z - 160 \geq 40 \quad \Rightarrow \quad Z \geq 50.$$

Let $\tilde{Z}$ be a Gaussian random variable approximating Z with

$$\mu_Z = np = 80(0.55) = 44, \quad \sigma_Z^2 = np(1-p) = 80(0.55)(0.45) = 19.8, \quad \sigma_Z = \sqrt{19.8} \approx 4.449.$$

Using the Gaussian approximation with continuity correction:

$$P(X \geq 40) = P(Z \geq 50) \approx P(\tilde{Z} \geq 49.5) = P\left(\frac{\tilde{Z} - 44}{4.449} \geq \frac{49.5 - 44}{4.449}\right) = P(Z' \geq 1.236).$$

From standard normal tables,

$$P(Z' \geq 1.236) = Q(1.236) = 0.108.$$

Therefore,

$$P(X \geq 40) \approx 0.108.$$

(c) Express the event $\{X = 0\}$ in terms of Z, and use the Gaussian approximation with continuity correction to estimate $P(X = 0)$.
Solution: The event $\{X = 0\}$ corresponds to

$$4Z - 160 = 0 \quad \Rightarrow \quad Z = 40.$$

Using continuity correction,

$$P(X = 0) = P(Z = 40) \approx P(39.5 \leq \tilde{Z} \leq 40.5).$$

Standardizing:

$$P(39.5 \leq \tilde{Z} \leq 40.5) = P\left(\frac{39.5 - 44}{4.449} \leq Z' \leq \frac{40.5 - 44}{4.449}\right) = P(-1.012 \leq Z' \leq -0.787).$$

Using the standard normal CDF:

$$P(X = 0) = \Phi(-0.787) - \Phi(-1.012) = (1 - \Phi(0.787)) - (1 - \Phi(1.012)) = \Phi(1.012) - \Phi(0.787).$$

From normal tables,

$$\Phi(1.012) = 0.8447, \quad \Phi(0.787) = 0.7849,$$

so

$$P(X = 0) \approx 0.8447 - 0.7849 = 0.0598.$$

2. **[Functions of Random Variables]**
Consider $Y = |X|$ where $X \sim \mathcal{N}(1, 4)$. Answer the following:

(a) Find an expression for the pdf of Y, $f_Y(v)$. Confirm that your pdf is correct by proving that it integrates to 1 over $\mathbb{R}$.

Solution: Since Y is $|X|$, it is a non-negative random variable, i.e., $Y > 0$. Starting with the CDF of Y with $v > 0$:

$$\begin{aligned}
P\{Y \le v\} &= P\{|X| \le v\} = P\{-v \le X \le v\} \\
&= P\left\{\frac{-v-1}{2} \le \frac{X-1}{2} \le \frac{v-1}{2}\right\} \\
&= \Phi\left(\frac{v-1}{2}\right) - \Phi\left(\frac{-v-1}{2}\right)
\end{aligned}$$

Taking the derivative w.r.t. v, we get:

$$\begin{aligned}
f_Y(v) &= \Phi'\left(\frac{v-1}{2}\right) \times \frac{1}{2} - \Phi'\left(\frac{-v-1}{2}\right) \times \frac{-1}{2} \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}} \times e^{\frac{-(v-1)^2}{8}} + \frac{1}{\sqrt{2\pi}} \times e^{\frac{-(v+1)^2}{8}}\right] = \mathcal{N}(1,4) + \mathcal{N}(-1,4)
\end{aligned}$$

Therefore,

$$f_Y(v) = \begin{cases} \frac{1}{\sqrt{8\pi}} \times e^{\frac{-(v-1)^2}{8}} + \frac{1}{\sqrt{8\pi}} \times e^{\frac{-(v+1)^2}{8}}, & \text{for } v \ge 0 \\ 0, & \text{for } v < 0 \end{cases}$$

(b) Use $f_Y(v)$ from Part (a) to calculate the probability $P\{Y > 4\}$.

Solution: Note that for $v \ge 0$, $f_Y(v) = \mathcal{N}(1,4) + \mathcal{N}(-1,4)$, i.e., the pdf of Y is the sum of two Gaussians. Therefore,

$$\begin{aligned}
P\{Y > 4\} &= \int_4^\infty f_Y(v)dv \\
&= \int_4^\infty [\mathcal{N}(1,4) + \mathcal{N}(-1,4)]dv \\
&= \int_{\frac{3}{2}}^\infty \mathcal{N}(0,1)dv + \int_{\frac{5}{2}}^\infty \mathcal{N}(0,1)dv = Q(1.5) + Q(2.5) \\
&\approx 0.0668 + 0.0062 = 0.0730
\end{aligned}$$

(c) Use the pdf of X, $f_X(u)$, to directly calculate the probability $P\{Y > 4\}$. Confirm that your answer is the same as in Part (b).

Solution:

$$\begin{aligned}
P\{Y > 4\} &= P\{|X| > 4\} = P\{\{X > 4\} \cup \{X < -4\}\} \\
&= P\{\{X > 4\} + P\{\{X < -4\} = P\{\{\frac{X-1}{2} > \frac{3}{2}\} + P\{\{\frac{X-1}{2} < -\frac{5}{2}\} \\
&= Q(1.5) + \Phi(-2.5) = Q(1.5) + Q(2.5) \\
&\approx 0.0668 + 0.0062 = 0.0730
\end{aligned}$$

which is the same as in Part (b).

3. **[Maximum Likelihood Estimation I]**
Random variable Y is defined as $Y = aX + b$. Answer the following:

(a) If $X \sim \text{Unif}[0,1]$ and $Y = 3$ is observed, find the maximum likelihood estimate $\hat{a}_{\text{ML}}$ and the density function $f_Y(v)$ in terms of b. (Assume $0 \le b \le 3$)

Solution: From the scaling rule, the likelihood function $f_\theta(v) = f_Y(v)$ with $\theta = a$ is given by:

$$
\begin{aligned}
f_\theta(v) &= \frac{1}{|\theta|} f_X\left(\frac{v-b}{\theta}\right) = \frac{1}{|\theta|}\mathbb{I}_{\{0 \le \frac{v-b}{\theta} \le 1\}} \\
f_\theta(3) &= \frac{1}{|\theta|}\mathbb{I}_{\{0 \le \frac{3-b}{\theta} \le 1\}}
\end{aligned}
$$

Where $\mathbb{I}$ is the indicator function–it evaluates to 1 when its condition is true, and 0 otherwise. The likelihood function $f_\theta(3)$ is maximized when $|\theta|$ is minimized while ensuring the indicator function evaluates to 1. Therefore, θ needs to satisfy:

$$
0 \le \frac{3-b}{\theta} \le 1 \implies 3 - b \le \theta
$$

Hence, $\hat{a}_{\text{ML}} = 3 - b$. Since $Y = \hat{a}_{\text{ML}} X + 1$, $Y \sim [\text{b}, 3]$ and

$$
f_Y(v) = \frac{1}{3-b}\mathbb{I}_{\{b < u \le 3\}}
$$

Alternatively, you can write this in piecewise form

$$
f_Y(v) = \begin{cases} \frac{1}{3-b} & b < v \le 3 \\ 0 & \text{otherwise} \end{cases}
$$

Alternative Approach: Since we have observed $Y = 3$ and we want to maximize $f_Y(3)$, we need $f_Y(3) > 0$, which indicates that $aX + b = 3$ is possible. As the support of X is $[0,1]$, $a \ge 3 - b$. With $f_Y(v) = \frac{1}{|a|} \cdot f_X(\frac{v-b}{a})$, we know $f_Y(3) = \frac{1}{|a|}$ under the condition $a \ge 3 - b$, and thus $\hat{a}_{\text{ML}} = 3 - b$ since it maximizes $\frac{1}{|a|}$. We can finally get $f_Y(v)$ by plugging in $a = \hat{a}_{\text{ML}}$.

(b) If $X \sim Exp(\lambda = 1)$ and $Y = 3$ is observed, find the maximum likelihood estimate $\hat{a}_{\text{ML}}$ and the density function $f_Y(v)$ in terms of b. (Assume $0 \le b < 3$)

Solution: From the scaling rule, the likelihood function $f_\theta(v) = f_Y(v)$ with $\theta = a$ is given by:

$$
\begin{aligned}
f_\theta(v) &= \frac{1}{|\theta|} f_X\left(\frac{v-b}{\theta}\right) = \frac{1}{|\theta|} e^{-\frac{v-b}{\theta}} \mathbb{I}_{\{\frac{v-b}{\theta} \ge 0\}} \\
f_\theta(3) &= \frac{1}{|\theta|} e^{-\frac{3-b}{\theta}} \mathbb{I}_{\{\frac{3-b}{\theta} \ge 0\}}
\end{aligned}
$$

Since $b < 3$, for $\theta > 0$ the log-likelihood is

$$
l(\theta) = log f_\theta(3) = -log\theta - \frac{3-b}{\theta}
$$

Taking the derivative and set to zero yields

$$
\begin{aligned}
l'(\theta) = -\frac{1}{\theta} + \frac{3-b}{\theta^2} &= 0 \\
\theta^* &= 3 - b
\end{aligned}
$$

The extrema happens at $\theta^* = 3 - b$. The second derivative $l''(3-b)$ is negative, so $\hat{a}_{\text{ML}} = 3 - b$ is the maximum. Since $Y = \hat{a}_{\text{ML}} X + 1$,

$$f_Y(v) = \frac{1}{3-b} e^{-\frac{v-b}{3-b}} \mathbb{I}_{\{v \geq b\}}$$

Alternatively, you can write this in piecewise form

$$f_Y(v) = \begin{cases} \frac{1}{3-b} e^{-\frac{v-b}{3-b}} & v \geq b \\ 0 & \text{otherwise} \end{cases}$$