

ECE 313: Problem Set 6: Problems and Solutions

Due: Friday, October 17 at 07:00:00 p.m.

Reading: *ECE 313 Course Notes*, Section 2.11.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: You must upload handwritten homework to Gradescope. No typeset homework will be accepted. No late homework will be accepted. Please write on the top right corner of the first page:

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SECTION

PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings.

1. [Hypothesis Testing for BO7]

Team A and team B are playing a game series of Best-of-7. This time, we are modeling the games with their history data instead of a simple win rate p for the team. Let X_k denotes the total games played in the BO7 game series won by team k . From team A 's matching history, they are good at blitz combat and the distribution of the games played across all won BO7s X_A follows $P_{X_A}(k) = [0.5, 0.3, 0.1, 0.1]$ for $k = [4, 5, 6, 7]$ respectively. On the other hand, team B is well-known for its toughness and resilience. Its winning distribution X_B follows $P_{X_B}(k) = [0.1, 0.2, 0.3, 0.4]$ for $k = [4, 5, 6, 7]$.

- (a) Let observation X denotes the total number of games being played for the BO7 game between A and B . We need to predict the winner between A and B using X . Construct the likelihood matrix, and find the ML decision rule.

Solution: The likelihood matrix for this problem is:

	$X = 4$	$X = 5$	$X = 6$	$X = 7$
A wins	<u>0.5</u>	<u>0.3</u>	0.1	0.1
B wins	0.1	0.2	<u>0.3</u>	<u>0.4</u>

The underlines in the table indicate which hypothesis is chosen.

- (b) Confirm that you obtain the same ML rule from the likelihood ratio form.

Solution: The likelihood ratio is given by

$$\Lambda(k) = \frac{p_A(k)}{p_B(k)} = [5, 1.5, 0.33, 0.25], \quad k = 4, 5, 6, 7.$$

The ML rule is therefore to declare A wins whenever $\Lambda(X) > 1$. We declare B wins when $\Lambda(X) < 1$. Note that $\Lambda(X)$ is not equal to 1, and therefore we do not need to break ties.

- (c) If we define A wins as the positive hypothesis H_1 , find $p_{\text{false-alarm}}$ and p_{miss} for the ML rule.

Solution:

$$p_{\text{false-alarm}} = P(\text{Claim } A \text{ wins} | B \text{ wins}) = 0.1 + 0.2 = 0.3,$$

and

$$p_{\text{miss}} = P(\text{Claim } B \text{ wins} | A \text{ wins}) = 0.1 + 0.1 = 0.2$$

- (d) According to the match history, team A keeps a 30% win rate on BO7 against team B . Find the average probability of error p_e for the ML rule.

Solution: In this case, we have prior $P(H_1) = \pi_1 = 0.3$ and $P(H_0) = \pi_0 = 0.7$

$$p_e = \pi_0 \cdot p_{\text{false-alarm}} + \pi_1 \cdot p_{\text{miss}} = 0.7 \times 0.3 + 0.3 \times 0.2 = 0.27.$$

- (e) Find the MAP decision rule using the joint probability matrix.

Solution: The joint probability matrix for this problem is:

	$X = 4$	$X = 5$	$X = 6$	$X = 7$
A wins	<u>0.15</u>	0.09	0.03	0.03
B wins	0.07	<u>0.14</u>	<u>0.21</u>	<u>0.28</u>

The underlines in the table indicate which hypothesis is chosen.

- (f) Confirm that you obtain the same MAP rule using the likelihood ratio form.

Solution: The MAP rule is to declare H_1 whenever

$$\Lambda(X) > \frac{\pi_0}{\pi_1} = \frac{7}{3}$$

or equivalently. We declare H_1 only when $X = 4, \Lambda(4) = 5 > \frac{7}{3}$.

- (g) Find the average probability of error p_e for the MAP rule.

Solution: Sum all the unchosen probabilities

$$p_e = 0.07 + 0.09 + 0.03 + 0.03 = 0.22$$

2. [Hypothesis Testing for Binomial Classification]

We are modeling a binary classification problem where the feature variable X represents the number of successes in a Binomial trial with $n = 2$. The two hypotheses correspond to two classes:

- H_1 : Class C_1 , where $X \sim \text{Binomial}(n = 2, p = 0.3)$
- H_0 : Class C_2 , where $X \sim \text{Binomial}(n = 2, p = 0.7)$

The prior probabilities are:

$$P(H_1) = \pi_1 = 0.2, \quad P(H_0) = \pi_0 = 0.8$$

Let $X \in \{0, 1, 2\}$ be the observed number of successes. We aim to predict the class label using X .

- (a) Construct the likelihood matrix using the Binomial PMF and find the ML decision rule.

Solution: We use the Binomial PMF:

$$P(X = x | H_i) = \binom{2}{x} p_i^x (1 - p_i)^{2-x}$$

For H_1 ($p = 0.3$):

$$P(X = 0 \mid H_1) = \binom{2}{0} (0.3)^0 (0.7)^2 = 1 \cdot 1 \cdot 0.49 = 0.49$$

$$P(X = 1 \mid H_1) = \binom{2}{1} (0.3)^1 (0.7)^1 = 2 \cdot 0.3 \cdot 0.7 = 0.42$$

$$P(X = 2 \mid H_1) = \binom{2}{2} (0.3)^2 (0.7)^0 = 1 \cdot 0.09 \cdot 1 = 0.09$$

For H_0 ($p = 0.7$):

$$P(X = 0 \mid H_0) = \binom{2}{0} (0.7)^0 (0.3)^2 = 1 \cdot 1 \cdot 0.09 = 0.09$$

$$P(X = 1 \mid H_0) = \binom{2}{1} (0.7)^1 (0.3)^1 = 2 \cdot 0.7 \cdot 0.3 = 0.42$$

$$P(X = 2 \mid H_0) = \binom{2}{2} (0.7)^2 (0.3)^0 = 1 \cdot 0.49 \cdot 1 = 0.49$$

Likelihood Matrix:

	$X = 0$	$X = 1$	$X = 2$
H_1	<u>0.49</u>	<u>0.42</u>	0.09
H_0	0.09	0.42	<u>0.49</u>

The underlines indicate the ML decision: choose the hypothesis with the higher likelihood. For $X = 1$, the likelihoods are equal, so we break the tie arbitrarily in favor of H_1 .

- (b) Confirm that you obtain the same ML rule from the likelihood ratio form.

Solution: The likelihood ratio is:

$$\Lambda(x) = \frac{P(X = x \mid H_1)}{P(X = x \mid H_0)} = [5.44, 1.00, 0.18], \quad x = 0, 1, 2$$

The ML rule is to declare H_1 whenever $\Lambda(X) > 1$, and H_0 when $\Lambda(X) < 1$. Since $\Lambda(1) = 1$, we break the tie arbitrarily in favor of H_1 . This matches the decisions in the likelihood matrix.

- (c) If we define H_1 (Class C_1) as the positive hypothesis, find $p_{\text{false-alarm}}$ and p_{miss} for the ML rule.

Solution:

$$p_{\text{false-alarm}} = P(\text{Claim } H_1 \mid H_0) = P(X = 0 \mid H_0) + P(X = 1 \mid H_0) = 0.09 + 0.42 = 0.51$$

$$p_{\text{miss}} = P(\text{Claim } H_0 \mid H_1) = P(X = 2 \mid H_1) = 0.09$$

- (d) Find the average probability of error p_e for the ML rule.

Solution: Using the priors $\pi_0 = 0.3$, $\pi_1 = 0.7$, we compute:

$$p_e = \pi_0 \cdot p_{\text{false-alarm}} + \pi_1 \cdot p_{\text{miss}} = 0.3 \cdot 0.51 + 0.7 \cdot 0.09 = 0.426$$

- (e) Find the MAP decision rule using the joint probability matrix.

Solution: We compute $P(X = x | H_i) \cdot P(H_i)$ for each x :

	$X = 0$	$X = 1$	$X = 2$
H_1	<u>0.098</u>	0.084	0.018
H_0	0.072	<u>0.336</u>	<u>0.392</u>

The underlines indicate the chosen hypothesis under the MAP rule.

- (f) Confirm that you obtain the same MAP rule using the likelihood ratio form.

Solution: The MAP rule is to declare H_1 whenever:

$$\Lambda(X) > \frac{\pi_0}{\pi_1} = \frac{0.8}{0.2} = 4$$

From earlier:

$$\Lambda(x) = [5.44, 1.00, 0.18]$$

So we declare H_1 for $X = 0$ and H_0 for $X = 1, 2$, which matches the MAP decisions using the joint probability matrix.

- (g) Find the average probability of error p_e for the MAP rule.

Solution: Sum all the unchosen joint probabilities:

$$p_e = P(H_0, X = 0) + P(H_1, X = 1) + P(H_1, X = 2) = 0.072 + 0.084 + 0.018 = 0.174$$

Note that the MAP rule gives a smaller probability of error as compared to the ML rule.

3. [ML testing for Radar]

Ben, a radar operator, is scanning a part of the sky to detect the presence of aircraft. The radar receiver comprises: 1) a sampler which captures samples of the analog received signal $x(t)$ every T seconds; followed by 2) a two-level slicer/quantizer that generates a binary output $X_i \in \{0, 1\}$ for the i^{th} sample; and 3) an accumulator that sums n consecutive binary outputs of the slicer to generate the observation X .

In presence (absence) of an aircraft, a '1' in the slicer output occurs with probability ρ_1 (ρ_0) with $0 < \rho_0 < \rho_1 < 1$. Let H_1 (H_0) denote the hypothesis that an aircraft is present (absent). Ben needs to determine which hypothesis is true based on the observation X over an observation interval of nT seconds. Assume now that an aircraft is three times as likely to be present in the part of the sky Ben is scanning.

In this problem Ben uses the ML decision rule.

- (a) Find the ML decision rule as a function of n , ρ_0 , and ρ_1 . Assume that ties are broken in favor of H_1 . Recall that $0 < \rho_0 < \rho_1 < 1$, and note that it is easier to use the likelihood ratio test approach.

Solution:

Each slicer output $X_i \in \{0, 1\}$ is a Bernoulli random variable. Under H_0 (no aircraft), $P(X_i = 1) = \rho_0$, and under H_1 (aircraft present), $P(X_i = 1) = \rho_1$, where $0 < \rho_0 < \rho_1 < 1$. The accumulator sums n consecutive samples:

$$X = \sum_{i=1}^n X_i.$$

Since the X_i are i.i.d. Bernoulli random variables, X follows a Binomial distribution:

$$X \sim \begin{cases} \text{Binomial}(n, \rho_0), & H_0, \\ \text{Binomial}(n, \rho_1), & H_1. \end{cases}$$

We can write the likelihoods as:

$$\mathcal{L}_0(x) = \binom{n}{x} \rho_0^x (1 - \rho_0)^{n-x}, \quad \mathcal{L}_1(x) = \binom{n}{x} \rho_1^x (1 - \rho_1)^{n-x}.$$

Now the ML rule tells us to choose the hypothesis that maximizes the likelihood. Equivalently, use the likelihood ratio test:

$$\frac{\mathcal{L}_1(X)}{\mathcal{L}_0(X)} \underset{H_0}{\overset{H_1}{\geq}} 1.$$

The Binomial coefficient cancels, yielding

$$\left(\frac{\rho_1}{\rho_0}\right)^X \left(\frac{1 - \rho_1}{1 - \rho_0}\right)^{n-X} \underset{H_0}{\overset{H_1}{\geq}} 1.$$

Taking logarithms gives

$$X \ln \frac{\rho_1(1 - \rho_0)}{\rho_0(1 - \rho_1)} \underset{H_0}{\overset{H_1}{\geq}} n \ln \frac{1 - \rho_0}{1 - \rho_1}.$$

Because $\rho_1 > \rho_0$, the coefficient of X on the left is positive, so this test is equivalent to comparing X with a threshold γ :

$$X \underset{H_0}{\overset{H_1}{\geq}} \gamma, \quad \gamma = \frac{n \ln\left(\frac{1 - \rho_0}{1 - \rho_1}\right)}{\ln\left(\frac{\rho_1(1 - \rho_0)}{\rho_0(1 - \rho_1)}\right)},$$

where ties are broken in favor of H_1 . Therefore the ML rule declares H_1 whenever

$$X \geq \frac{n \ln\left(\frac{1 - \rho_0}{1 - \rho_1}\right)}{\ln\left(\frac{\rho_1(1 - \rho_0)}{\rho_0(1 - \rho_1)}\right)},$$

and declares H_0 otherwise.

- (b) Find $p_{\text{false-alarm}}$, p_{miss} , and p_e for the ML rule, assuming that $n = 8$, $\rho_0 = 0.25$ and $\rho_1 = 0.75$.

Solution: For the choice of n , ρ_0 and ρ_1 ,

$$\ln\left(\frac{\rho_1}{\rho_0}\right) = \ln\left(\frac{1 - \rho_0}{1 - \rho_1}\right) = \ln 3.$$

Hence, the ML rule declares H_1 whenever $X \geq 4$, and declares H_0 whenever $X \leq 3$. And

$$\begin{aligned} p_{\text{false-alarm}} &= P\{X \geq 4 | H_0 \text{ true}\} = \sum_{k=4}^8 \binom{8}{k} (0.25)^k (0.75)^{8-k} \\ &= 0.114, \end{aligned}$$

and

$$\begin{aligned} p_{\text{miss}} &= P\{X \leq 3 | H_1 \text{ true}\} \\ &= \sum_{k=0}^3 \binom{8}{k} (0.75)^k (0.25)^{8-k} = 0.027. \end{aligned}$$

Since $\pi_1 = 3\pi_0 \implies \pi_0 = 0.25, \pi_1 = 0.75$, therefore

$$p_e = p_{\text{false-alarm}} \times \pi_0 + p_{\text{miss}} \times \pi_1 = 0.114 * 0.25 + 0.027 * 0.75 = 0.0475.$$

4. [MAP testing for Radar]

Consider the same hypothesis testing problem as in Problem 3.

- (a) Find the MAP decision rule as a function of n , ρ_0 , and ρ_1 . Assume that ties are broken in favor of H_1 . Again note that it is easier to use the likelihood ratio test approach.

Solution: As before, the accumulator output is

$$X = \sum_{i=1}^n X_i, \quad \begin{cases} H_0 : X \sim \text{Binomial}(n, \rho_0), \\ H_1 : X \sim \text{Binomial}(n, \rho_1), \end{cases} \quad 0 < \rho_0 < \rho_1 < 1.$$

Let the priors be $\pi_0 = P(H_0)$ and $\pi_1 = P(H_1)$. The problem statement says the aircraft is three times as likely to be present, so $\pi_1/\pi_0 = 3$.

The MAP rule compares posteriors:

$$\frac{\mathcal{L}_1(X)}{\mathcal{L}_0(X)} \frac{\pi_1}{\pi_0} \underset{H_0}{\overset{H_1}{\geq}} 1 \iff \frac{\mathcal{L}_1(X)}{\mathcal{L}_0(X)} \underset{H_0}{\overset{H_1}{\geq}} \frac{\pi_0}{\pi_1}.$$

With Binomial likelihoods,

$$\left(\frac{\rho_1}{\rho_0}\right)^X \left(\frac{1-\rho_1}{1-\rho_0}\right)^{n-X} \underset{H_0}{\overset{H_1}{\geq}} \frac{\pi_0}{\pi_1}.$$

Taking logs and collecting the X terms,

$$X \left[\ln \frac{\rho_1(1-\rho_0)}{\rho_0(1-\rho_1)} \right] \underset{H_0}{\overset{H_1}{\geq}} n \ln \frac{1-\rho_0}{1-\rho_1} + \ln \frac{\pi_0}{\pi_1}.$$

Because $\rho_1 > \rho_0$, the bracketed coefficient is positive, so the test is a threshold on X :

$$X \underset{H_0}{\overset{H_1}{\geq}} \gamma_{\text{MAP}}, \quad \gamma_{\text{MAP}} = \frac{n \ln\left(\frac{1-\rho_0}{1-\rho_1}\right) + \ln\left(\frac{\pi_0}{\pi_1}\right)}{\ln\left(\frac{\rho_1(1-\rho_0)}{\rho_0(1-\rho_1)}\right)}.$$

with ties broken in favor of H_1 .

Here $\ln(\pi_0/\pi_1) = \ln(1/3) = -\ln 3$, so

$$\gamma_{\text{MAP}} = \frac{n \ln\left(\frac{1-\rho_0}{1-\rho_1}\right) - \ln 3}{\ln\left(\frac{\rho_1(1-\rho_0)}{\rho_0(1-\rho_1)}\right)}.$$

As expected, the prior favoring H_1 lowers the threshold relative to ML.

- (b) Find the average probability of error p_e for the MAP rule, assuming that $n = 8$, $\rho_0 = 0.25$ and $\rho_1 = 0.75$.

Solution:

For $n = 8$, $\rho_0 = 0.25$, $\rho_1 = 0.75$, and $\pi_1/\pi_0 = 3$, we have

$$\gamma_{\text{MAP}} = 3.5 \Rightarrow \text{decide } H_1 \text{ iff } X \geq 4 \text{ (ties in favor of } H_1).$$

Thus, in this case, the MAP decision rule and the ML decision rule in Problem 3 turn out to be identical. Hence, their probability of error, probability of false alarm, and probability of miss will also be identical. Thus, $p_{\text{false-alarm}} = 0.114$, $p_{\text{miss}} = 0.027$, and $p_e = 0.0475$ for the MAP rule as well. Note: as this problem illustrates, the MAP and ML decision rules *can* turn out to be identical even for unequal priors, i.e., $\pi_1 \neq \pi_0$. The two rules are *guaranteed* to be identical when the priors are equal.

5. [Inequalities in Probability]

Let $\{A_1, A_2, \dots, A_n\}$ be a collection of arbitrary events. Determine whether the following inequality holds:

$$P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i) - \sum_{2 \leq i \leq n} P(A_i \cap A_1).$$

Solution: Base case ($n = 2$). We have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2),$$

which matches the claimed inequality with equality.

Inductive step. Assume the claim holds for some $n - 1 \geq 2$, i.e.,

$$P\left(\bigcup_{i=1}^{n-1} A_i\right) \leq \sum_{i=1}^{n-1} P(A_i) - \sum_{i=2}^{n-1} P(A_i \cap A_1).$$

Consider $U_n := \bigcup_{i=1}^n A_i = \left(\bigcup_{i=1}^{n-1} A_i\right) \cup A_n$. Using the two-set identity,

$$P(U_n) = P\left(\bigcup_{i=1}^{n-1} A_i\right) + P(A_n) - P\left(A_n \cap \bigcup_{i=1}^{n-1} A_i\right).$$

Apply the inductive hypothesis to the first term and note that

$$P\left(A_n \cap \bigcup_{i=1}^{n-1} A_i\right) \geq P(A_n \cap A_1),$$

since $A_1 \subseteq \bigcup_{i=1}^{n-1} A_i$. Therefore,

$$P(U_n) \leq \left[\sum_{i=1}^{n-1} P(A_i) - \sum_{i=2}^{n-1} P(A_i \cap A_1) \right] + P(A_n) - P(A_n \cap A_1),$$

which simplifies to

$$P\left(\bigcup_{i=1}^n A_i\right) \leq \sum_{i=1}^n P(A_i) - \sum_{i=2}^n P(A_i \cap A_1).$$

This completes the induction.