University of Illinois **Fall 2025**

ECE 313: Problem Set 6

Due: Friday, October 17 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Section 2.11.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: You must upload handwritten homework to Gradescope. No typeset homework will be accepted. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.

1. **[Hypothesis Testing for BO7]**
   Team A and team B are playing a game series of Best-of-7. This time, we are modeling the games with their history data instead of a simple win rate p for the team. Let X_k denotes the total games played in the BO7 game series won by team k. From team A's matching history, they are good at blitz combat and the distribution of the games played across all won BO7s X_A follows $P_{X_A}(k) = [0.5, 0.3, 0.1, 0.1]$ for $k = [4, 5, 6, 7]$ respectively. On the other hand, team B is well-known for its toughness and resilience. Its winning distribution X_B follows $P_{X_B}(k) = [0.1, 0.2, 0.3, 0.4]$ for $k = [4, 5, 6, 7]$.

   (a) Let observation X denotes the total number of games being played for the BO7 game betwen A and B. We need to predict the winner between A and B using X. Construct the likelihood matrix, and find the ML decision rule.

   (b) Confirm that you obtain the same ML rule from the likelihood ratio form.

   (c) If we define A wins as the positive hypothesis H_1, find $p_{\text{false-alarm}}$ and p_{miss} for the ML rule.

   (d) According to the match history, team A keeps a 30% win rate on BO7 against team B. Find the average probability of error p_e for the ML rule.

   (e) Find the MAP decision rule using the joint probability matrix.

   (f) Confirm that you obtain the same MAP rule using the likelihood ratio form.

   (g) Find the average probability of error p_e for the MAP rule.

2. **[Hypothesis Testing for Binomial Classification]**
   We are modeling a binary classification problem where the feature variable X represents the number of successes in a Binomial trial with $n = 2$. The two hypotheses correspond to two classes:

   - H_1: Class C_1, where $X \sim \text{Binomial}(n = 2, p = 0.3)$
   - H_0: Class C_2, where $X \sim \text{Binomial}(n = 2, p = 0.7)$

The prior probabilities are:

$$P(H_1) = \pi_1 = 0.2, \quad P(H_0) = \pi_0 = 0.8$$

Let $X \in \{0, 1, 2\}$ be the observed number of successes. We aim to predict the class label using X.

(a) Construct the likelihood matrix using the Binomial PMF and find the ML decision rule.

(b) Confirm that you obtain the same ML rule from the likelihood ratio form.

(c) If we define H_1 (Class C_1) as the positive hypothesis, find $p_{\text{false-alarm}}$ and p_{miss} for the ML rule.

(d) Find the average probability of error p_e for the ML rule.

(e) Find the MAP decision rule using the joint probability matrix.

(f) Confirm that you obtain the same MAP rule using the likelihood ratio form.

(g) Find the average probability of error p_e for the MAP rule.

3. **[ML testing for Radar]**
Ben, a radar operator, is scanning a part of the sky to detect the presence of aircraft. The radar receiver comprises: 1) a sampler which captures samples of the analog received signal $x(t)$ every T seconds; followed by 2) a two-level slicer/quantizer that generates a binary output $X_i \in \{0, 1\}$ for the i^{th} sample; and 3) an accumulator that sums n consecutive binary outputs of the slicer to generate the observation X.

In presence (absence) of an aircraft, a '1' in the slicer output occurs with probability ρ_1 (ρ_0) with $0 < \rho_0 < \rho_1 < 1$. Let H_1 (H_0) denote the hypothesis that an aircraft is present (absent). Ben needs to determine which hypothesis is true based on the observation X over an observation interval of nT seconds. Assume now that an aircraft is three times as likely to be present in the part of the sky Ben is scanning.

In this problem Ben uses the ML decision rule.

(a) Find the ML decision rule as a function of n, ρ_0, and ρ_1. Assume that ties are broken in favor of H_1. Recall that $0 < \rho_0 < \rho_1 < 1$, and note that it is easier to use the likelihood ratio test approach.

(b) Find $p_{\text{false-alarm}}$, p_{miss}, and p_e for the ML rule, assuming that $n = 8$, $\rho_0 = 0.25$ and $\rho_1 = 0.75$.

4. **[MAP testing for Radar]**
Consider the same hypothesis testing problem as in Problem 3.

(a) Find the MAP decision rule as a function of n, ρ_0, and ρ_1. Assume that ties are broken in favor of H_1. Again note that it is easier to use the likelihood ratio test approach.

(b) Find the average probability of error p_e for the MAP rule, assuming that $n = 8$, $\rho_0 = 0.25$ and $\rho_1 = 0.75$.

5. **[Inequalities in Probability]**
Let $\{A_1, A_2, \ldots, A_n\}$ be a collection of arbitrary events. Determine whether the following inequality holds:

$$P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) - \sum_{2 \leq i \leq n} P(A_i \cap A_1).$$