University of Illinois **Fall 2025**

ECE 313: Problem Set 5

Due: Sunday, October 12 at 07:00:00 p.m. **Note the later due date.**
Reading: *ECE 313 Course Notes,* Section 2.8, 2.9.

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.

1. [**A Variant of Geometric Distribution**]
   In class, we defined a geometric random variable (X_{Trials}) as the number of *trials* needed to get the first success in a sequence of independent Bernoulli trials with probability of success p. Another way of defining a geometric random variable (X_{Failures}) could be to count the number of *failures* before the first success. Do the following.

   (a) Find the pmf of X_{Failures} and verfiy that it is indeed a pmf.

   (b) Express X_{Failures} in terms of X_{Trials}. Calculate the mean and variance of X_{Failures}.

   (c) Calculate the maximum likelihood estimate of p based on a single observation from X_{Failures}. How does it compare to the maximum likelihood estimate of p based on a single observation from X_{Trials}?

2. [**Maximum Likelihood Parameter Estimation**]
   A biased coin when tossed shows a Heads with probability p and Tails with probability $1-p$.

   (a) The biased coin is tossed 10 times, and 6 Heads are observed. What is the maximum likelihood estimate $\hat{p}_{\text{ML}}$ of p given this observation?

   (b) Suppose it is known that $p = 0.05$. The biased coin is now tossed an unknown number n times during which 6 Heads are observed. What is the maximum likelihood estimate $\hat{n}_{\text{ML}}$ of n given this observation?

3. [**Markov Inequality**]
   Let X denote the outcome of rolling a fair die. We define two random variables $Y = X^2$ and $Z = X^2 - 15$.

   (a) Find $E[Y]$ and $E[Z]$

   (b) Find the exact probability for $\{Y \geq c\}$ as a function of c for $c \in \{1, 10, 100\}$, and verify if the Markov inequality holds for all these c.

(c) Find the exact probability for $\{Z \geq c\}$ as a function of c for $c \in \{1, 10, 100\}$, and verify if the Markov inequality holds for all these c. Why does Markov inequality not hold for Z?

4. **[Chip Testing]**
Alice is a graduate student who has designed an integrated circuit (IC) implementing an machine learning accelerator IC in a 45 nm semiconductor process as part of her graduate research. She has just received 50 packaged chips and is getting ready to test them to see if it is working properly. Alice wants to show that her design can classify images with high accuracy p_a. To do that she tests her chip with n images and counts the number E that are incorrectly classified. She obtains an accuracy estimate $\hat{p}_a = 1 - \frac{E}{n}$. Alice hopes to write-up a research paper on her design and submit it to ISSCC, a top circuits conference. All she needs is one working chip that classifies images from the test set with high accuracy in order to report the results (yield is not an issue in papers from academia). However, testing a chip is a slow process and Alice wants to minimize the testing time so she can submit the paper before the deadline.

   (a) Determine the probability distribution of the random variable E representing the misclassification error count.

   (b) Alice tests the first chip using $n = 100$ test images and finds that 95 images are correctly classified. Is it ok for Alice to report that her design gives an accuracy of $p_a = 0.95$? Give reasons.

   (c) How many test vectors should Alice test her chip with so that she can report that the true accuracy of her design p_a lies in the interval $\hat{p}_a \pm 1\%$ with a confidence level greater than 95%?

   (d) Since testing is a slow process, Alice decides to do a quick pass through all of her 50 chips by testing with 2500 images to find "good parts" which she will then test with large number of vectors as in Part (c). She would still like to achieve a high confidence level of 96%? What confidence interval can she achieve?

5. **[Airline industries]**
Each airplane has capacity for 150 passengers, and overbooking is a common practice in these industries.

   (a) To sell more tickets than available seats, the airline needs to estimate the probability that each passenger will attend the flight. Suppose that each passenger will attend the flight with probability p. The airline uses $\hat{p}_n = X/n$ as the estimate of p, where n is the number of sold tickets and X is the number of people who attended the flight. How large n should be to estimate p within 0.1 with confidence of 0.99?

   (b) According to the historical data, each passenger will attend a flight with probability $p_{\text{attend}} = 0.9$. What is the maximum number of tickets the airline can sell to ensure that no one is left behind with probability 0.75? (Hint: Use Chebyshev's inequality, roots of $0.9x^2 + 0.6x - 150 = 0$ are -13.25 and 12.58.)

6. **[Discrete Random Variable on Even Integers]**
Let X denote a discrete random variable that takes on even integer values $0, 2, 4, \ldots, n$, and zero otherwise.

(a) Let the pmf of X be given by

$$p_X(k) = \frac{3(2^k)}{4(2^n) - 1}, \quad \text{for even integer values } k \in \{0, 2, 4, \dots, n\},$$

where the value of n is unknown. Find the maximum-likelihood estimate $\hat{n}_{\mathrm{ML}}$ from the observation that $X = 10$ on a trial of the experiment.

(b) Now, let

$$p_X(k) = a, \quad \text{for even integer values } k \in \{0, 2, 4, \dots, n\},$$

and zero otherwise. Find the constant a that makes this a valid pmf and compute its mean.