University of Illinois **Fall 2025**

ECE 313: Problem Set 4: Problems and Solutions

Due: Friday, Oct 3rd at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.4-2.7

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. [**Packet Transmission**]
   Alice is communicating with Bob on a link. The bit error rate (BER) of a link is defined as the probability of a bit being incorrectly decoded by Bob. Bit errors are independent.

   (a) Determine the probability distribution of the random variable A representing the number of CORRECT bits in a packet of size 1000 bits. Assume the BER of the link is 10^{-6}.

   Solution: The correct count A can be obtained by running $n = 1000$ independent trials of a Bernoulli random variable with parameter $p_a = 1 - 10^{-6}$. Thus, A is a binomial random variable with parameters $(n = 1000, p_a = 0.999999)$, i.e., $A \sim \text{Binomial}(1000, 0.999999)$.

   (b) A packet error occurs if any of the bits in the packet is incorrectly decoded. If Alice wants the packet error rate to be no greater than 0.1%, what is the maximum packet size in bits she can transmit? Assume the BER of the link is 10^{-6}.

   Solution: For a k-bit packet, the packet error rate $p_E = 1 - 0.999999^k$. Solving $\arg\max_k p_E < 0.001$ s.t. $k \in \mathbb{N}$, which gives $k^* = 1000$ bits.

   (c) Now consider BER of the link is 0.001. Alice and Bob now use an error correction scheme that can detect and correct up to 2 bit errors per packet. What is the packet error rate of a 1000-bit packet?

   Solution: The packet will be correct if there are 0, 1, or 2 bits in error. $p_E = 1 - (0.999)^{1000} - \binom{1000}{1}(0.999)^{999}(0.001) - \binom{1000}{2}(0.999)^{998}(0.001)^2 = 0.0802$. Packet error rate is around 8.02%

   (d) Suppose Alice transmits a 10^{10} bits (10 Gb) file to Bob on a link with BER $= 10^{-9}$. What is the probability that exactly 5 bits are decoded incorrectly? Please approximate the answer using natural exponent e.

Solution: Since n is large and p is small, we can approximate the Binomial with Poisson distribution X of mean $\lambda = np = 10^{10} \times 10^{-9} = 10$. The probability of exactly 5 bits being wrong is $p_X(5) \approx \frac{\lambda^5 e^{-\lambda}}{5!} = \frac{10^5}{5!} e^{-10}$.

2. **[Best of K]**
   Two teams A and B are playing a "Best of 7" (BO7) series: whoever wins the 4 games will win and end the series. Assume there are no ties each game. For each game, team A has a probability 0.6 to win and team B has a probability of 0.4 to win. Team B already won the first game. Let X be the total number of games played.

   (a) What is $p_X(5)$? That is, exactly 5 games are played in the series, including the first game won by B.

   Solution: X represents the total number of games played until one of the teams wins including the first game won by B. Therefore, the pmf of X is given by:

$$P\{X = 5\} = P\{S_{4,p} = 4\} + P\{S_{3,1-p} = 4\}$$

   where $S_{r,p} \sim \text{NB}(r, p)$. In the equation above, the first (second) term represents A (B) winning the series. Setting $p = 0.6$, we get

$$P\{X = 5\} = \binom{3}{3}(0.6)^4 + \binom{3}{2}(0.4)^3 \times (0.6)^1 = 0.2448.$$

   (b) Given we know that all 7 games are played in the series, what's the probability that team A wins the series? Please write down the equations using conditional probability.

   Solution:

$$P\{A \text{ wins}|X = 7\} = \frac{P\{A \text{ wins}, X = 7\}}{P\{X = 7, A \text{ wins}\} + P\{X = 7, B \text{ wins}\}}$$

$$P\{X = 7, A \text{ wins}\} = P\{S_{4,p} = 6\} = \binom{5}{3}(0.6)^4 \times (0.4)^2 \approx 0.207$$

$$P\{X = 7, B \text{ wins}\} = P\{S_{3,1-p} = 6\} = \binom{5}{2}(0.4)^3 \times (0.6)^3 \approx 0.138.$$

   There the final solution is written as

$$P\{A \text{ wins}|X = 7\} = \frac{\binom{5}{3}(0.6)^4(0.4)^2}{\binom{5}{3}(0.6)^4(0.4)^2 + \binom{5}{2}(0.6)^3(0.4)^3} = 0.6.$$

   (c) Assume team A can not bear long gaming. The win rate of team A will drop to 0.3 in games 6 and 7. What is the probability of exactly 6 games being played in this series?

   Solution: X is no longer a sum of two NB() RVs. Therefore, using first principles:

$$P\{X = 6, A \text{ wins}\} = \binom{4}{3}(0.6)^3(0.4)(0.3) \approx 0.104,$$

   which means A must score 3 wins from game 2 to game 5, and wins game 6. We also have

$$P\{X = 6, B \text{ wins}\} = \binom{4}{2}(0.6)^2(0.4)^2(0.7) \approx 0.242$$

   Here B wins 2 out of 4 games from game 2 to 5, and wins game 6. We finally have $p_X(6) = 4 \times (0.6)^3(0.4)(0.3) + 6 \times (0.6)^2(0.4)^2(0.7) = 0.346$.

3. **[Automatic Repeat Request (ARQ)]**

The GPU A in a datacenter communicates with GPU B via a link. The link transmits data at a rate of $r = 200$ Gbps and makes independent bit errors with probability $p = 10^{-3}$. To transmit data, GPU A constructs frames of length $n = 1000$-bits in which $k = 950$-bits are payload and $n - k = 50$ bits are cyclic redundancy check (CRC) bits to enable GPU B to detect frame errors defined as having at least one bit error in a frame. On detecting a frame error, GPU B sends a retransmission request (NCK) to GPU A, otherwise it sends an ACK. The round-trip delay from GPU A transmitting a frame and receiving an ACK/NCK for it is $N = 10$ frame duration long.

In the following, let $X = 1$ ($X = 0$) if a frame is received correctly (in error). Let Y be the number of transmissions per frame. The presence of CRC and retransmissions reduces the effective data rate from r to

$$r_{\text{eff}} = r \times \frac{k}{n} \times \frac{1}{E[Y]}.$$

Now answer the following:

(a) What is the probability of correct frame transmission $p_{\text{nrt}} = P\{X = 1\}$? What type of an RV is X?

Solution: Since there are n-bits in a frame and each can flip independently with probability p, the probability of receiving a correct frame is $p_{\text{nrt}} = (1 - p)^n = (0.999)^{1000} = 0.367$. Thus, $X \sim \text{Bernoulli}(p_{\text{nrt}} = 0.367)$.

(b) In ARQ1 protocol, GPU A keeps transmitting frames as the ACKs are received and retransmits only the frame for which it received a NCK. Derive an expression for the average number of transmissions per frame and the effective data rate r_{eff} in terms of r, n, k and p.

Solution: No retransmission is needed when a frame is received correctly. This event occurs with probability p_{nrt}. Treating frame transmission as a coin toss with success probability p_{nrt}, we find that Y (number of transmissions per frame) to be a $\text{Geom}(p_{\text{nrt}})$ RV. Thus, the average number of transmissions per frame is $E[Y] = \frac{1}{p_{\text{nrt}}} = 2.72$. Furthermore, in every n-bit frame there are k payload bits. Thus, the effective data rate is:

$$r_{\text{eff}} = r \left(\frac{k}{n}\right) \frac{1}{E[Y]} = r \left(\frac{k}{n}\right) p_{\text{nrt}} = 69.73 \text{ Gbps}$$

Thus, the effective data rate has dropped from 200 Gbps to 69.73 Gbps due to retransmission and CRC overhead.

(c) In ARQ2 protocol, GPU A retransmits *not only* the frame for which it received a NCK (lets call it frame i) but also the $N - 1 = 9$ frames that were transmitted during the time between the first transmission of frame i and the reception of NCK. Derive an expression for the average number of retransmissions per frame and the effective data rate r_{eff} in terms of r, n, k, N, and p.

Solution: In this case, each NCK results in N frames being retransmitted including the erroneously received frame i. Thus, Y takes values $1, N + 1, 2N + 1, ...$, with a pmf $p_Y(1) = p_{\text{nrt}}$, $p_Y(N + 1) = (1 - p_{\text{nrt}})p_{\text{nrt}}$,...,$p_Y(kN + 1) = (1 - p_{\text{nrt}})^k p_{\text{nrt}}$.
Hence, one can express $Y = (Z - 1)N + 1$ where $Z \sim \text{Geom}(p_{\text{nrt}})$, and

$$E[Y] = E[(Z - 1)N + 1] = 1 + NE[Z - 1] = 1 + N(\frac{1}{p_{\text{nrt}}} - 1) = \frac{p_{\text{nrt}} + N(1 - p_{\text{nrt}})}{p_{\text{nrt}}}.$$

Thus, $E[Y] = 18.24$, and

$$r_{\text{eff}} = r\left(\frac{k}{n}\right)\frac{1}{E[Y]} = 10.41\text{Gbps}.$$

As seen ARQ2 protocol has a more severe penalty than ARQ1.

(d) Both ARQ1 and ARQ2 protocols need to storage buffers in GPU A and GPU B in order to ensure that GPU A retransmits frames until received correctly by GPU B and GPU B is able to release the frames in order. Which of the two protocols do you think will need fewer buffers? Provide a qualitative answer with reasons.

Solution: ARQ1 protocol is more complex than ARQ2 since GPU B needs to store the $N-1$ correctly received frames until the first NCKed frame is correctly received. This is the price for its bandwidth efficiency.

4. **[Martingale Gambling Strategy]**
The Martingale gambling strategy is a famous betting system often used in coin-toss gambling. A gambler repeatedly bets on a game of win rate p and payout odds 2 (a win pays profit equal to your stake). The rules are:

- The gambler starts by betting \$1.
- If they win, they stop immediately, take their winnings (net profit = \$1), and leave the game.
- If they lose, they double the bet (so the next bet is \$2, then \$4, etc.) and continue until the first win.

The idea is that the first win always recovers all previous losses and yields a net gain of exactly \$1. Let X denote the total number of games played.

(a) Suppose the gambler has infinite bankroll, i.e., they will play until the win. Compute the probability of $X = k$ for any given k and $E[X]$.

Solution: $X \sim \text{Geom}(p)$. We have $p_X(k) = (1-p)^{k-1}p$ and $E[X] = 1/p$.

(b) Suppose that the gambler has only a finite bankroll of \$4095. What is the probability that the gambler win at least \$1?

Solution: The total bet gambler places until game K is $B_{total} = \sum_{k=0}^{K-1} 2^k = 2^K - 1$. For $B_{total} = 4095$, we have the gambler can play at most $K = 12$ games. The probability of getting a win in 12 games or less is $P\{X \leq 12\} = 1 - (1-p)^{12}$.

(c) If $p > \frac{1}{2}$, compute the expected size of the final bet (the one you win with). The answer should be a closed form scalar, not infinite geometric series.

Solution: For game k, the final bet is $B_{final} = 2^{k-1}$. The expectation can be solved as the sum of a geometric series.

$$\begin{aligned} E[B_{total}] &= \sum_{k=1}^{\infty} 2^{k-1}(1-p)^{k-1}p \\ &= \frac{p}{1-2(1-p)} \\ &= \frac{p}{2p-1}. \end{aligned}$$

The expectation is only positive when $p > \frac{1}{2}$.

(d) Now assume the gambler wants to win \$3 instead of \$1. To do so, the gambler is allowed to make bets until the third win. Let Y denotes the total number of games being played. Find the pmf and variance of Y.

Solution: $Y \sim \text{NB}(3, p)$ follows negative binomial distribution with parameter $(3, p)$. The pmf and variance are as follows:

$$P_Y(k) = \binom{k-1}{2} p^3 (1-p)^{k-3},$$

$$Var(Y) = \frac{3(1-p)}{p^2}.$$

5. **[Binomial and Poisson]**
   Answer the following questions:

(a) Let $X \sim \text{Bin}(n, p)$. Find $P(X \text{ is even})$ in terms of n and p.
(Hint: Compute $P(X \text{ is even}) - P(X \text{ is odd})$)

Solution: Let $q = 1-p$. We can use the binomial series to help us solve $P(X \text{ is even}) + P(X \text{ is odd})$ and $P(X \text{ is even}) - P(X \text{ is odd})$. We have

$$\begin{aligned}
&P(X \text{ is even}) + P(X \text{ is odd}) \\
&= \binom{n}{0} p^0 q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \binom{n}{3} p^3 q^{n-3} + \dots \\
&= (q+p)^n = 1
\end{aligned}$$

and

$$\begin{aligned}
&P(X \text{ is even}) - P(X \text{ is odd}) \\
&= \binom{n}{0} p^0 q^n - \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} - \binom{n}{3} p^3 q^{n-3} + \dots \\
&= \binom{n}{0} (-p)^0 q^n + \binom{n}{1} (-p)^1 q^{n-1} + \binom{n}{2} (-p)^2 q^{n-2} + \binom{n}{3} (-p)^3 q^{n-3} + \dots \\
&= (q-p)^n
\end{aligned}$$

Therefore,

$$P(X \text{ is even}) = \frac{(q+p)^n + (q-p)^n}{2} = \frac{1 + (1-2p)^n}{2}$$

(b) Let $Y \sim \text{Poi}(\lambda)$. Find $P(Y \text{ is even})$ in terms of λ.

Solution: We have

$$\begin{aligned}
P(Y \text{ is even}) + P(Y \text{ is odd}) &= e^{-\lambda}\frac{\lambda^0}{0!} + e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!} + e^{-\lambda}\frac{\lambda^3}{3!} + \dots \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} e^{\lambda} = 1
\end{aligned}$$

and

$$
\begin{aligned}
P(Y \text{ is even}) - P(Y \text{ is odd}) &= e^{-\lambda}\frac{\lambda^0}{0!} - e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!} - e^{-\lambda}\frac{\lambda^3}{3!} + \dots \\
&= e^{-\lambda}\frac{(-\lambda)^0}{0!} + e^{-\lambda}\frac{(-\lambda)^1}{1!} + e^{-\lambda}\frac{(-\lambda)^2}{2!} + e^{-\lambda}\frac{(-\lambda)^3}{3!} + \dots \\
&= e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} \\
&= e^{-\lambda}e^{-\lambda} = e^{-2\lambda}
\end{aligned}
$$

Therefore,

$$
P(Y \text{ is even}) = \frac{1+e^{-2\lambda}}{2}
$$

(c) Suppose that $n \to \infty$ and $p \to 0$ such that $np = \lambda$. Verify that your answer in part (a) converges to the answer in part (b).

Solution: From Euler relationship (see (2.9) in the course notes),

$$
(1-\frac{\lambda}{n})^n \to e^{-\lambda} \text{ as } n \to \infty, np = \lambda
$$

Therefore,

$$
\begin{aligned}
P(X \text{ is even}) &= \frac{1+(1-2p)^n}{2} \\
&= \frac{1+(1-\frac{2\lambda}{n})^n}{2} \\
&\to \frac{1+e^{-2\lambda}}{2} \\
&= P(Y \text{ is even}) \text{ as } n \to \infty
\end{aligned}
$$