University of Illinois **Fall 2025**

ECE 313: Problem Set 3: Problems and Solutions

Due: Friday, September 26 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.3, 2.10, 2.4.1-2.4.2

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. [**The Biased Coin Game**]
   You are playing a game with two coins:

   - **Coin A** is fair: it has a probability of $\frac{1}{2}$ of landing heads.
   - **Coin B** is biased: it has a probability of $\frac{3}{4}$ of landing heads.

   One of the coins is randomly selected (with equal probability), and then flipped three times. You observe the following outcome:

   Two heads and one tail

   What is the probability that the coin used was **Coin B**, given the observed outcome?

   Solution: Let C_A and C_B denote the events that Coin A and Coin B were chosen, respectively. Let E be the event that the outcome is two heads and one tail.

   We want to compute:

   $$P(C_B \mid E) = \frac{P(E \mid C_B) \cdot P(C_B)}{P(E)}$$

   Since the coin is chosen at random:

   $$P(C_A) = P(C_B) = \frac{1}{2}$$

   Compute the likelihoods:

$$P(E \mid C_A) = \binom{3}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right) = 3 \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{8}$$

$$P(E \mid C_B) = \binom{3}{2} \cdot \left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{4}\right) = 3 \cdot \frac{9}{16} \cdot \frac{1}{4} = \frac{27}{64}$$

Now compute the total probability of observing two heads and one tail:

$$P(E) = P(E \mid C_A) \cdot P(C_A) + P(E \mid C_B) \cdot P(C_B)$$

$$P(E) = \frac{3}{8} \cdot \frac{1}{2} + \frac{27}{64} \cdot \frac{1}{2} = \frac{3}{16} + \frac{27}{128} = \frac{24}{128} + \frac{27}{128} = \frac{51}{128}$$

Finally, apply Bayes' Theorem:

$$P(C_B \mid E) = \frac{\frac{27}{64} \cdot \frac{1}{2}}{\frac{51}{128}} = \frac{\frac{27}{128}}{\frac{51}{128}} = \frac{27}{51} = \frac{9}{17}$$

2. **[Mutually Exclusive and Independent Events]**
   Prove that two events with positive probabilities cannot simultaneously be mutually exclusive and mutually independent.

   Solution: Let A, B be events.

   If A and B are independent, then

   $$P(A \cap B) = P(A)P(B).$$

   Since $P(A) > 0$ and $P(B) > 0$, it follows that $P(A \cap B) > 0$. Thus, A and B cannot be mutually exclusive (which would require $P(A \cap B) = 0$).

   If A and B are mutually exclusive, then

   $$P(A \cap B) = 0.$$

   But independence would require

   $$P(A \cap B) = P(A)P(B) > 0,$$

   since both probabilities are positive. This is a contradiction. Therefore, mutually exclusive events with positive probability cannot be independent.

3. **[Monty Hall Problem]**
   In a game show, a contestant is presented with three doors. Behind one door is a car, and behind the other two doors are goats. The contestant selects Door 1. ats, the host selects one at random.

   In this instance, the host opens Door 3, revealing a goat. The contestant is then offered the chance to switch their choice to Door 2.

(a) Use Bayes Theorem to compute the probability that the car is behind Door 2, given that the host opened Door 3 to reveal a goat.

Solution: Let C_i denote the event that the car is behind Door i ($i = 1, 2, 3$). Let H_3 denote the event that the host opens Door 3.

We want:

$$P(C_2 \mid H_3) = \frac{P(H_3 \mid C_2)\,P(C_2)}{P(H_3)}.$$

Since the car is equally likely to be behind any door:

$$P(C_1) = P(C_2) = P(C_3) = \tfrac{1}{3}.$$

As the contestant selects Door 1, if the car is behind Door 2, the host must open Door 3, therefore:

$$P(H_3 \mid C_2) = 1$$

We can also obtain:

$$P(H_3 \mid C_1) = \tfrac{1}{2} \quad \text{(host opens Door 2 or 3 at random)},$$

$$P(H_3 \mid C_3) = 0 \quad \text{(host would not open the car door)}.$$

The total probability of H_3 is computed by:

$$P(H_3) = P(H_3 \mid C_1) \cdot P(C_1) + P(H_3 \mid C_2) \cdot P(C_2) + P(H_3 \mid C_3) \cdot P(C_3) = \tfrac{1}{2} \cdot \tfrac{1}{3} + 1 \cdot \tfrac{1}{3} + 0 \cdot \tfrac{1}{3} = \tfrac{1}{6} + \tfrac{1}{3} = \tfrac{1}{2}.$$

Finally, apply Bayes Theorem:

$$P(C_2 \mid H_3) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

(b) Based on your calculation, should the contestant switch or stay with their original choice (Door 1)? Justify your answer using the probabilities you computed.

Solution: If the contestant stays with Door 1, the probability of winning is

$$P(C_1 \mid H_3) = \frac{P(H_3 \mid C_1)\,P(C_1)}{P(H_3)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \tfrac{1}{3}.$$

If the contestant switches to Door 2, the probability of winning is

$$P(C_2 \mid H_3) = \tfrac{2}{3}.$$

Therefore, the contestant should switch.

Another solution, albeit longer, to parts (a) and (b) is to enumerate the sample space Ω which is given by:

$$\Omega = \{(i, j, k) : i, j, k \in \{1, 2, 3\}, k \neq i, k \neq j\}$$

where i is the door behind which there is a car (car door), j is the door chosen by the contestant, and k is the door opened by the host.

The cardinality $|\Omega| = 3 \times 2 \times 1 + 3 \times 1 \times 2 = 12$ where the first term is the number of outcomes in which the contestant does not choose the car door, and the second term is the number of ways in which the contestant does choose it. Enumerating all 12 outcomes below:

$$\begin{aligned}\Omega =\{&(1,1,2),(1,1,3),(1,2,3),(1,3,2),\\ &(2,2,1),(2,2,3),(2,1,3),(2,3,1),\\ &(3,3,1),(3,3,2),(3,1,2),(3,2,1)\}\end{aligned}$$

However, the outcomes are not equally likely, e.g.,

$$P\{(1,1,2)\} = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{18}; P\{(1,1,3)\} = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{18};$$

$$P\{(1,2,3)\} = \frac{1}{3} \times \frac{1}{3} \times 1 = \frac{1}{9}; P\{(1,3,2)\} = \frac{1}{3} \times \frac{1}{3} \times 1 = \frac{1}{9}$$

i.e., singleton events with outcomes having $i = j$ occur with probability $1/18$ otherwise they occur with probability $1/9$.

Now, define event A as "contestant chooses door 1 and host opens door 3". Then, $A = \{(1,1,3),(2,1,3)\}$. Also, define C_1 as the event that car is behind door 1. Thus,

$$P\{C_1|A\} = \frac{P\{(1,1,3)\}}{P\{(1,1,3)\} + P\{(2,1,3)\}} = \frac{\frac{1}{18}}{\frac{1}{18}+\frac{1}{9}} = \frac{1}{3}$$

Similarly,

$$P\{C_2|A\} = \frac{P\{(2,1,3)\}}{P\{(1,1,3)\} + P\{(2,1,3)\}} = \frac{\frac{1}{9}}{\frac{1}{18}+\frac{1}{9}} = \frac{2}{3}$$

The use of Bayes rule is simpler but it requires a proper definition of the events of interest. The direct method is longer but provides good intuition.

4. **[Medical Diagnostics]**
A certain disease affects 2% of a population. A diagnostic test is used to detect the disease. The test has the following properties:

- If a person **has** the disease, the test returns positive with probability 0.95 (true positive rate).
- If a person **does not** have the disease, the test returns positive with probability 0.10 (false positive rate).

Let D be the event that a randomly selected person has the disease, and T be the event that the test result is positive.

(a) Compute the following probabilities: $P(D)$, $P(T \mid D)$, $P(T \mid D^c)$, and $P(T)$.
Solution:

$$P(D) = 0.02, \quad P(D^c) = 0.98, \quad P(T \mid D) = 0.95, \quad P(T \mid D^c) = 0.10.$$

$$P(T) = P(T \mid D)P(D) + P(T \mid D^c)P(D^c) = 0.95 \cdot 0.02 + 0.10 \cdot 0.98 = 0.117.$$

(b) Compute the probability that a person has the disease given a positive test.

Solution: Apply Bayes' Theorem:

$$P(D \mid T) = \frac{P(T \mid D)P(D)}{P(T)} = \frac{0.95 \cdot 0.02}{0.117} = \frac{0.019}{0.117} = \frac{19}{117} \approx 0.1624.$$

(c) Determine whether the events D and T are independent.

Solution: They are not independent since

$$P(D \mid T) = \frac{19}{117} \neq P(D) = 0.02$$

(d) Suppose a second, independent test is administered. Let T_2 be the event that the second test is positive. Assume the second test has the same accuracy as the first and is conditionally independent of the first test, given the disease status. Compute: $P(D \mid T \cap T_2)$.

Solution: Conditional independence given disease status gives

$$P(T \cap T_2 \mid D) = 0.95^2 = 0.9025, \qquad P(T \cap T_2 \mid D^c) = 0.10^2 = 0.01.$$

Using Law of Total probability:

$$P(T \cap T_2) = 0.9025 \cdot 0.02 + 0.01 \cdot 0.98 = 0.02785.$$

Apply Bayes' Theorem:

$$P(D \mid T \cap T_2) = \frac{0.9025 \cdot 0.02}{0.02785} = \frac{0.01805}{0.02785} = \frac{361}{557} \approx 0.6481.$$

5. [**More on Throwing Dice**]

Two fair dice are thrown. Let E denote the event that the sum of the dice is 7. Let F denote the event that the first die equals 4 and let G be the event that the second die equals 3.

(a) Are E and F independent events? Are E and G independent events?

Solution: There are 6 outcomes where E is satisfied: (1,6),(2,5),(3,4), and their opposites. Therefore $P(E) = \frac{1}{6}$. Also, $P(F) = \frac{1}{6}$. The intersection between these two events is one outcome, (4,3). Therefore because $P(E \cap F) = P(E)P(F)$, the two events are independent.

Like for F, there are 6 outcomes where G is satisfied. The intersection between E and G is again (4,3). Therefore because $P(E \cap G) = P(E)P(G)$, the two events are independent.

(b) Are E and $F \cap G$ independent events?

Solution: Another way to show independence is to show that $P(E|F \cap G) = P(E)$. If we know that F and G are both satisfied, we know that the outcome must have been (4,3). So $P(E|\text{the roll was (4,3)})$ is obviously 1. However, $P(E) = \frac{1}{36}$ as we previously showed. So E and $F \cap G$ are not independent.

(c) Are the events E, F and G mutually independent events?

Solution: Mutual independence of three events, E, F and G, requires all pairs, namely, E and F, F and G, and E and G to be independent and that $P(E \cap F \cap G) = P(E)P(F)P(G)$. From (a), note that we have all pairs to be independent. But, from (c), we have $P(E \cap F \cap G) \neq P(E)P(F)P(G)$. Hence, E, F, and G are not mutually independent.

6. **[Reliable Bit Transmission with Majority Voting]**
In a digital communication system, each **data bit** is transmitted multiple times to reduce the impact of noise. Suppose each copy of a bit is independently corrupted with probability $p = 0.1$ during transmission.

To improve reliability, the system uses **redundant transmission**: each data bit is sent **10 times**. The receiver applies **majority voting**, which means it decides the original bit based on the value that appears *more than half* the time among the received copies.

Let $X \sim \text{Binomial}(n = 10, p = 0.1)$, where X is the number of corrupted copies of a single bit.

(a) What is the probability that the receiver correctly detects the original bit using majority voting?

Solution: Majority voting is correct if $X \leq 4$, i.e., 4 or fewer corrupted copies.

$$P(X \leq 4) = \sum_{k=0}^{4} \binom{10}{k} (0.1)^k (0.9)^{10-k} \approx 0.998$$

So, the probability of correct detection is approximately 99.8%.

(b) Suppose the system is upgraded and now each bit is sent **15 times**. What is the minimum number of corrupted copies that would cause an incorrect detection?

Solution: Majority voting fails if more than half the copies are corrupted.

$$\left\lfloor \frac{15}{2} \right\rfloor + 1 = 8$$

Therefore, if 8 or more copies are corrupted, the majority vote will be incorrect.

(c) For a general number of repetitions n, derive a formula for the probability of correct detection assuming corruption probability p and majority voting.

Solution: Let $X \sim \text{Binomial}(n, p)$. Majority voting is correct if:

$$X \leq \left\lfloor \frac{n-1}{2} \right\rfloor$$

So the probability of correct detection is:

$$P_{\text{correct}}(n, p) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{k} p^k (1-p)^{n-k}$$

(d) If the receiver observes that in 1000 transmitted bits, the majority vote was incorrect 120 times, estimate the corruption probability p using the binomial model.

Solution: Given: 1000 transmissions, 120 incorrect majority votes.

Estimate:

$$q = \frac{120}{1000} = 0.12$$

We want to find p such that:

$$P(X > 4) = 0.12 \quad \text{for } X \sim \text{Binomial}(10, p)$$

Using trial-and-error:

$$p \approx 0.28$$

So, the estimated corruption probability is approximately 28%.

7. **[Sum of Bernoulli Random Variables]**
Answer the following.

(a) Let $X_1, \ldots, X_n$ be n Bernoulli random variables each with parameter p. Consider a new random variable, $S_1 = X_1 + \cdots + X_n$. Calculate $E(S_1)$.
Solution:

$$\begin{aligned} E(S_1) &= E(X_1 + \cdots + X_n) \\ &= E(X_1) + \cdots + E(X_n) && \text{(Linearity of expectation)} \\ &= p + \cdots + p \\ &= np \end{aligned}$$

(b) Let S_2 be a Binomial random variable with parameters n and p. Calculate $E(S_2)$.
Solution: Refer to the book. $E(S_2) = np$

(c) Is $E(S_1) = E(S_2)$? If so, is it sufficient to conclude that S_1 is also a Binomial random variable with parameters n and p? Justify your answer.
Solution: Yes. No, just because the expectations of two random variables are the same, it does not mean they have the same distribution.