University of Illinois **Fall 2025**

ECE 313: Problem Set 3

Due: Friday, September 26 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.3, 2.10, 2.4.1-2.4.2

Note on reading: For most sections of the course notes, there are short-answer questions at the end of the chapter. We recommend that after reading each section, you try answering the short-answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write at the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. [**The Biased Coin Game**]
   You are playing a game with two coins:

   - **Coin A** is fair: it has a probability of $\frac{1}{2}$ of landing heads.
   - **Coin B** is biased: it has a probability of $\frac{3}{4}$ of landing heads.

   One of the coins is randomly selected (with equal probability), and then flipped three times. You observe the following outcome:

   Two heads and one tail

   What is the probability that the coin used was **Coin B**, given the observed outcome?

2. [**Mutually Exclusive and Independent Events**]
   Prove that two events with positive probabilities cannot simultaneously be mutually exclusive and mutually independent.

3. [**Monty Hall Problem**]
   In a game show, a contestant is presented with three doors. Behind one door is a car, and behind the other two doors are goats. The contestant selects Door 1.

   The host, who knows what is behind each door, then opens one of the two remaining doors—always choosing a door that was not selected by the contestant and that hides a goat. If both unchosen doors hide goats, the host selects one at random.

   In this instance, the host opens Door 3, revealing a goat. The contestant is then offered the chance to switch their choice to Door 2.

(a) Use Bayes' Theorem to compute the probability that the car is behind Door 2, given that the host opened Door 3 to reveal a goat.

(b) Based on your calculation, should the contestant switch or stay with their original choice (Door 1)? Justify your answer using the probabilities you computed.

4. **[Medical Diagnostics]**
A certain disease affects 2% of a population. A diagnostic test is used to detect the disease. The test has the following properties:

- If a person **has** the disease, the test returns positive with probability 0.95 (true positive rate).
- If a person **does not** have the disease, the test returns positive with probability 0.10 (false positive rate).

Let D be the event that a randomly selected person has the disease, and T be the event that the test result is positive.

(a) Compute the following probabilities: $P(D)$, $P(T \mid D)$, $P(T \mid D^c)$, and $P(T)$.

(b) Compute the probability that a person has the disease given a positive test.

(c) Determine whether the events D and T are independent.

(d) Suppose a second, independent test is administered. Let T_2 be the event that the second test is positive. Assume the second test has the same accuracy as the first and is conditionally independent of the first test, given the disease status. Compute: $P(D \mid T \cap T_2)$.

5. **[More on Throwing Dice]**
Two fair dice are thrown. Let E denote the event that the sum of the dice is 7. Let F denote the event that the first die equals 4 and let G be the event that the second die equals 3.

(a) Are E and F independent events? Are E and G independent events?

(b) Are E and $F \cap G$ independent events?

(c) Are the events E, F and G mutually independent events?

6. **[Reliable Bit Transmission with Majority Voting]**
In a digital communication system, each **data bit** is transmitted multiple times to reduce the impact of noise. Suppose each copy of a bit is independently corrupted with probability $p = 0.1$ during transmission.

To improve reliability, the system uses **redundant transmission**: each data bit is sent **10 times**. The receiver applies **majority voting**, which means it decides the original bit based on the value that appears *more than half* the time among the received copies.

Let $X \sim \text{Binomial}(n = 10, p = 0.1)$, where X is the number of corrupted copies of a single bit.

(a) What is the probability that the receiver correctly detects the original bit using majority voting?

(b) Suppose the system is upgraded and now each bit is sent **15 times**. What is the minimum number of corrupted copies that would cause an incorrect detection?

(c) For a general number of repetitions n, derive a formula for the probability of correct detection assuming corruption probability p and majority voting.

(d) If the receiver observes that in 1000 transmitted bits, the majority vote was incorrect 120 times, estimate the corruption probability p using the binomial model.

7. [**Sum of Bernoulli Random Variables**]
   Answer the following.

   (a) Let $X_1, \ldots, X_n$ be n Bernoulli random variables each with parameter p. Consider a new random variable, $S_1 = X_1 + \cdots + X_n$. Calculate $E(S_1)$.

   (b) Let S_2 be a Binomial random variable with parameters n and p. Calculate $E(S_2)$.

   (c) Is $E(S_1) = E(S_2)$? If so, is it sufficient to conclude that S_1 is also a Binomial random variable with parameters n and p? Justify your answer.