University of Illinois **Fall 2025**

ECE 313: Problem Set 2: Problems and Solutions

Due: Friday, September 19 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.1 - 2.4.2.

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. **[PMF, Mean, and Variance]**
   You roll a fair coin and a fair die simultaneously. Let $X = 1$ ($X = 0$) if the coin shows a head (tail) and Y equals the number showing on the die. Furthermore, let $Z = 2X + Y$.

   (a) Find the pmfs of X, Y, and Z.
   Solution: The coin is fair, so

$$p_X(x) = P\{X = x\} = \begin{cases} \frac{1}{2}, & x = 0, \\ \frac{1}{2}, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

   The die is also fair with outcomes {1,2,. . . ,6}, so

$$p_Y(y) = P\{Y = y\} = \begin{cases} \frac{1}{6}, & y \in \{1, 2, 3, 4, 5, 6\}, \\ 0, & \text{otherwise.} \end{cases}$$

   Possible values of Z:

   - If $X = 0$ then $Z = Y \in \{1, 2, 3, 4, 5, 6\}$.
   - If $X = 1$ then $Z = 2 + Y \in \{3, 4, 5, 6, 7, 8\}$.

   The sample space of this experiment is $\Omega = \{(i, j) : i \in \{0, 1\}, j \in \{1, 2, 3, 4, 5, 6\}\}$ with $|\Omega| = 2 \times 6 = 12$ and all outcomes being equally likely. Thus,

$$P\{X = i, Y = j\} = \frac{1}{12}$$

   Alternatively, since the coin toss and die roll are independent,

$$P\{X = x, Y = y\} = P\{X = x\}P\{Y = y\} = \tfrac{1}{2} \cdot \tfrac{1}{6} = \tfrac{1}{12} \quad \text{for } x \in \{0, 1\},\ y \in \{1, \dots, 6\}.$$

Thus, we can compute $p_Z(z) = P\{Z = z\}$ by summing over the pairs (x, y) such that $2x + y = z$.

Finally,

$$p_Z(z) = \begin{cases} \frac{1}{12}, & z = 1, 2, \\ \frac{1}{6}, & z = 3, 4, 5, 6, \\ \frac{1}{12}, & z = 7, 8, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Find $E[X]$, $E[Y]$, and $E[Z]$ using the pmfs of X, Y, and Z, respectively. Can you guess a general relationship between $E[X]$, $E[Y]$, and $E[Z]$?

Solution: Expectation of X:

$$E[X] = \sum_x x\, p_X(x) = 0 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{2} = \tfrac{1}{2}.$$

Expectation of Y:

$$E[Y] = \sum_{y=1}^{6} y\, p_Y(y) = \sum_{y=1}^{6} y \cdot \tfrac{1}{6} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}.$$

Expectation of Z:

$$E[Z] = \sum_z z\, p_Z(z).$$

From the pmf of Z:

$$E[Z] = 1 \cdot \frac{1}{12} + 2 \cdot \tfrac{1}{12} + (3+4+5+6) \cdot \frac{1}{6} + 7 \cdot \frac{1}{12} + 8 \cdot \frac{1}{12} = \frac{9}{2}.$$

Comparing the numerical values of $E[X]$, $E[Y]$, and $E[Z]$, we can guess that $E[Z] = 2E[X] + E[Y]$. This result will be proved later in the course.

(c) Find $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Var}(Z)$. Can you guess a general relationship between $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Var}(Z)$?

Solution: We have

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

Compute $E\left[X^2\right]$:

$$E\left[X^2\right] = \sum_x x^2\, p_X(x) = 1^2 \cdot \tfrac{1}{2} + 0^2 \cdot \tfrac{1}{2} = \tfrac{1}{2}.$$

So the variance is:

$$\mathrm{Var}(X) = \tfrac{1}{2} - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}.$$

You can also directly use the definition,

$$\mathrm{Var}(X) = E[(X - E[X])^2] = \sum_x (x - E[X])^2\, p_X(x).$$

Which finally gives us:

$$\mathrm{Var}(X) = (0 - \frac{1}{2})^2 \cdot \frac{1}{2} + (1 - \frac{1}{2})^2 \cdot \frac{1}{2} = \frac{1}{4}.$$

Variance of Y:

$$\mathrm{Var}(Y) = \sum_{y=1}^{6} (y - E[Y])^2 \, p_Y(y), \quad E[Y] = \frac{7}{2}.$$

Compute each term:

$$(1-3.5)^2 = 6.25, \quad (2-3.5)^2 = 2.25, \quad (3-3.5)^2 = 0.25,$$
$$(4-3.5)^2 = 0.25, \quad (5-3.5)^2 = 2.25, \quad (6-3.5)^2 = 6.25.$$

Multiply each by $p_Y(y) = 1/6$ and sum:

$$\mathrm{Var}(Y) = \frac{6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25}{6} = \frac{17.5}{6} = \frac{35}{12}.$$

Step 3. Variance of Z:

$$\mathrm{Var}(Z) = \sum_{z} (z - E[Z])^2 \, p_Z(z), \quad E[Z] = \frac{9}{2}.$$

Compute $(z-4.5)^2 \, p_Z(z)$ for each z:

$$\begin{aligned}
z = 1 &: (1-4.5)^2 \cdot \tfrac{1}{12} = 12.25 \cdot \tfrac{1}{12} = \tfrac{49}{48}, \\
z = 2 &: (2-4.5)^2 \cdot \tfrac{1}{12} = 6.25 \cdot \tfrac{1}{12} = \tfrac{25}{48}, \\
z = 3 &: (3-4.5)^2 \cdot \tfrac{1}{6} = 2.25 \cdot \tfrac{1}{6} = \tfrac{3}{8}, \\
z = 4 &: (4-4.5)^2 \cdot \tfrac{1}{6} = 0.25 \cdot \tfrac{1}{6} = \tfrac{1}{24}, \\
z = 5 &: (5-4.5)^2 \cdot \tfrac{1}{6} = 0.25 \cdot \tfrac{1}{6} = \tfrac{1}{24}, \\
z = 6 &: (6-4.5)^2 \cdot \tfrac{1}{6} = 2.25 \cdot \tfrac{1}{6} = \tfrac{3}{8}, \\
z = 7 &: (7-4.5)^2 \cdot \tfrac{1}{12} = 6.25 \cdot \tfrac{1}{12} = \tfrac{25}{48}, \\
z = 8 &: (8-4.5)^2 \cdot \tfrac{1}{12} = 12.25 \cdot \tfrac{1}{12} = \tfrac{49}{48}.
\end{aligned}$$

Add all terms using a common denominator 48:

$$\mathrm{Var}(Z) = \frac{49 + 25 + 18 + 2 + 2 + 18 + 25 + 49}{48} = \frac{188}{48} = \frac{47}{12}.$$

Comparing the numerical values of $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Var}(Z)$, we can guess that $\mathrm{Var}(Z) = 4\mathrm{Var}(X) + \mathrm{Var}(Y)$. This result will be proved later in the course.

(d) Find $E[1/Z]$ and $\mathrm{Var}(1/Z)$?

Solution: Compute $E[1/Z]$:

$$E\left[\frac{1}{Z}\right] = \sum_z \frac{1}{z} \, p_Z(z) = \frac{1}{12}\left(\frac{1}{1} + \frac{1}{2}\right) + \frac{1}{6}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) + \frac{1}{12}\left(\frac{1}{7} + \frac{1}{8}\right) \approx 0.306.$$

Compute $E[1/Z^2]$:

$$E\left[\frac{1}{Z^2}\right] = \sum_z \frac{1}{z^2} \, p_Z(z) = \frac{1}{12}\left(1^2 + \frac{1}{4}\right) + \frac{1}{6}\left(\frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}\right) + \frac{1}{12}\left(\frac{1}{49} + \frac{1}{64}\right) \approx 0.1474.$$

Compute $\mathrm{Var}(1/Z)$:

$$\mathrm{Var}\left(\frac{1}{Z}\right) = E\left[\frac{1}{Z^2}\right] - \left(E\left[\frac{1}{Z}\right]\right)^2 \approx 0.1474 - (0.306)^2 \approx 0.1474 - 0.0936 \approx 0.0538$$

2. **[PMF, Mean and standard deviation]**
Suppose two fair dice are rolled independently, so the sample space is $\Omega = \{(i,j) : 1 \le i \le 6,\ \text{and}\ 1 \le j \le 6\}$, and all outcomes are equally likely. Let X be the random variable defined by $X(i,j) = i - j$, and let Y be the random variable defined by $Y(i,j) = \max\{0, i-j\}$ (Rectified Linear Unit or ReLU).

(a) Derive the pmf of X and sketch it.
Solution: As $X(i,j) = i - j \in \{-5,-4,-3,-2,-1,0,1,2,3,4,5\}$, we note that there are 6 ways for $X = 0$ to occur, 5 ways for $X = 1$ or $X = -1$ to occur, and so on. Hence, $p_X(k) = \frac{6-|k|}{36}$.
Figure 1 is a graph of the pmf of X:

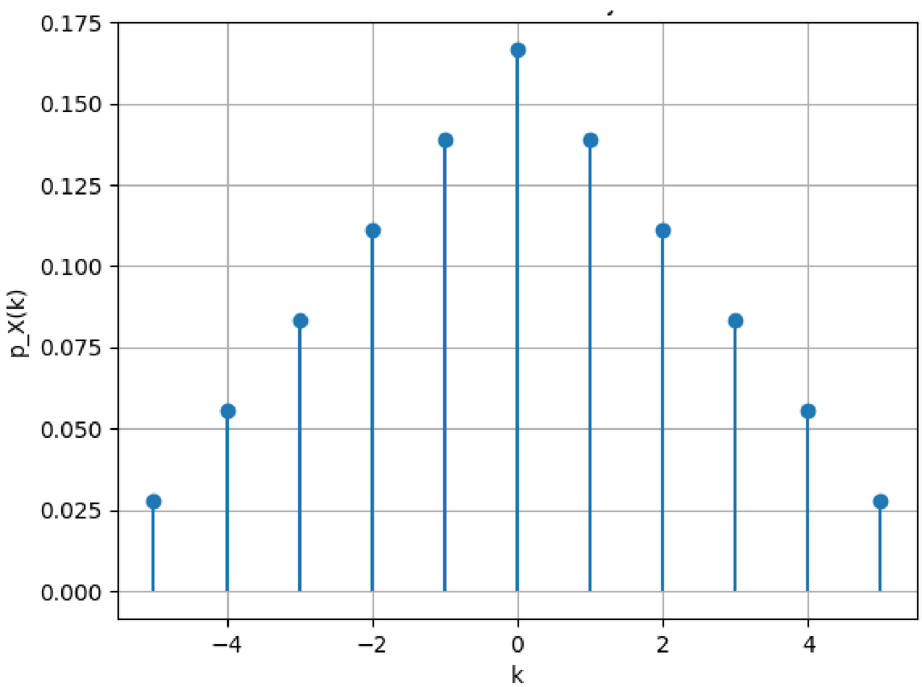

Figure 1: The pmf of $X(i,j) = i - j$.

(b) Find the mean, $E[X]$, and standard deviation, σ_X, of X.
Solution: Since the pmf of X is symmetric around $X = 0$ (see Fig. 1), $E[X] = 0$. But we can show this mathematically as shown below:

$$E[X] = 0 \cdot \frac{6}{36} + (1-1) \cdot \frac{5}{36} + (2-2) \cdot \frac{4}{36} + (3-3) \cdot \frac{3}{36} + (4-4) \cdot \frac{2}{36} \tag{1}$$

$$+ (5-5) \cdot \frac{1}{36} = 0 \tag{2}$$

$$E[X^2] = (0)^2 \cdot \frac{6}{36} + ((1)^2 + (-1)^2) \cdot \frac{5}{36} + ((2)^2 + (-2)^2) \cdot \frac{4}{36} + ((3)^2 + (-3)^2) \cdot \frac{3}{36} \tag{3}$$

$$+ ((4)^2 + (-4)^2) \cdot \frac{2}{36} + ((5)^2 + (-5)^2) \cdot \frac{1}{36} = \frac{210}{36} = 5.83 \tag{4}$$

$$\tag{5}$$

$$\sigma_X = \sqrt{\text{Var}(X)} = \sqrt{E[X^2] - E[X]^2} = \sqrt{E[X^2]} = \sqrt{\frac{210}{36}} \approx 2.42 \tag{6}$$

(c) Derive the pmf of Y and sketch it.

Solution: Y takes values in $\{0, 1, 2, 3, 4, 5\}$. Here $Y = 0$ occurs whenever $i - j \leq 0$, i.e., $i \leq j$. Also, $Y = k > 0$ occurs whenever $i - j = k$, i.e., $i = j + k$. We count the number of ways for $Y = k$ to occur:

- $Y = 0$ (i.e., $i \leq j$):

$$(i, j) = (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6), (4,4), (4,5), (4,6), (5,5), (5,6), (6,6)$$

- $Y = 1$ (i.e., $i - j = 1$): $(2,1), (3,2), (4,3), (5,4), (6,5)$
- $Y = 2$ (i.e., $i - j = 2$): $(3,1), (4,2), (5,3), (6,4)$
- $Y = 3$ (i.e., $i - j = 3$): $(4,1), (5,2), (6,3)$
- $Y = 4$ (i.e., $i - j = 4$): $(5,1), (6,2)$
- $Y = 5$ (i.e., $i - j = 5$): $(6,1)$

The pmf of Y is $p_Y(k) = P\{Y = k\}$, and for each case we have

$$\begin{aligned}
p_Y(0) &= \frac{21}{36} = \frac{7}{12}, \\
p_Y(1) &= \frac{5}{36}, \\
p_Y(2) &= \frac{4}{36} = \frac{1}{9}, \\
p_Y(3) &= \frac{3}{36} = \frac{1}{12}, \\
p_Y(4) &= \frac{2}{36} = \frac{1}{18}, \\
p_Y(5) &= \frac{1}{36}.
\end{aligned}$$

Another way to find $p_Y(k)$ is to realize that $Y = X$ when $Y > 0$ and

$$P\{Y = 0\} = P\{X \leq 0\} = \sum_{k=-5}^{0} p_X(k) = \frac{7}{12}$$

hence

$$p_Y(k) = \begin{cases} \frac{7}{12}, & k = 0 \\ p_X(k), & k = 1, 2, 3, 4, 5 \\ 0, & \text{otherwise.} \end{cases}$$

Figure 2 is a graph of the pmf of Y.

(d) Find the mean, $E[Y]$, and standard deviation, σ_Y, of Y.

Solution: Compute the mean:

$$E[Y] = \sum_{k=0}^{5} k\, p_Y(k) = 0 \cdot \tfrac{21}{36} + 1 \cdot \tfrac{5}{36} + 2 \cdot \tfrac{4}{36} + 3 \cdot \tfrac{3}{36} + 4 \cdot \tfrac{2}{36} + 5 \cdot \tfrac{1}{36} = \tfrac{35}{36}.$$

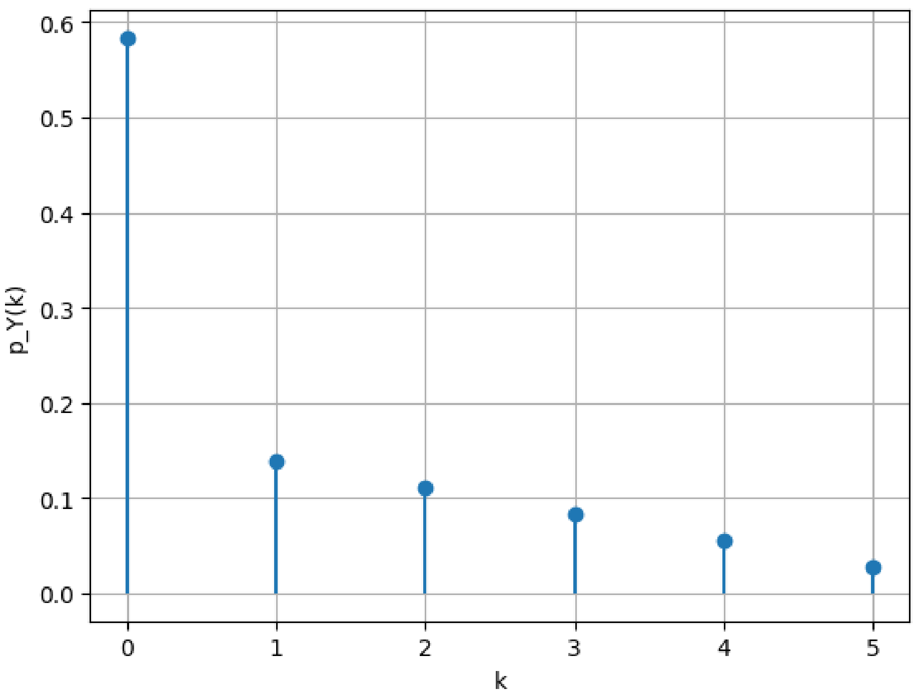

Figure 2: The pmf of $Y(i,j) = \max\{0, i-j\}$ (ReLU of $i-j$).

Compute the second moment:

$$E[Y^2] = \sum_{k=0}^{5} k^2\, p_Y(k) = 1^2 \cdot \tfrac{5}{36} + 2^2 \cdot \tfrac{4}{36} + 3^2 \cdot \tfrac{3}{36} + 4^2 \cdot \tfrac{2}{36} + 5^2 \cdot \tfrac{1}{36} = \tfrac{35}{12}.$$

Compute the variance:

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = \frac{35}{12} - \left(\frac{35}{36}\right)^2.$$

Standard deviation:

$$\sigma_Y = \sqrt{\mathrm{Var}(Y)} \approx 1.40.$$

3. **[PMF, Mean]**
Suppose that you as a UIUC student are planning on going to a concert in the Krannert Center. Ten of your friends are also interested in the same concert. Four of your friends are UIUC students, whereas the remaining six friends are not students. You randomly choose three of your friends to go with you. When you get to the Krannert Center, it turns out that UIUC students pay \$10 and those who are not students pay \$50 for their tickets. Let X be the total amount of money you and your friends end up paying to get into the concert.

   (a) What values can X take?
   Solution: For k non-students going with you, the total cost is $10 + 50(k) + 10(3 - k)$ (because you are also going). You can take between 0 and 3 non-students. $X \in \{40, 80, 120, 160\}$.

(b) Find the pmf of X.

Solution:

Suppose you choose 0 non-student friends. In this case, the total amount is 40.
$\# \; ways = \binom{4}{3} \cdot \binom{6}{0} = 4$
Suppose you choose 1 non-student friend. In this case, the total amount is 80.
$\# \; ways = \binom{4}{2} \cdot \binom{6}{1} = 36$
Suppose you choose 2 non-student friends. In this case you all have to pay 120.
$\# \; ways = \binom{4}{1} \cdot \binom{6}{2} = 60$
Suppose you choose 3 non-student friends. In this case, the total amount is 160.
$\# \; ways = \binom{4}{0} \cdot \binom{6}{3} = 20$
There are 120 total ways to pick people, and all are equally likely.

$$pmf_X(k) = \begin{cases} \frac{4}{120} & k = 40 \\ \frac{36}{120} & k = 80 \\ \frac{60}{120} & k = 120 \\ \frac{20}{120} & k = 160 \end{cases}$$

(c) Find the expected value of X.

Solution:

$$E[X] = 40 \cdot \frac{4}{120} + 80 \cdot \frac{36}{120} + 120 \cdot \frac{60}{120} + 160 \cdot \frac{20}{120} \quad (7)$$
$$= 112 \quad (8)$$

4. **[Matching cards to boxes]**

Three boxes are placed on a table, with the i-th box containing a card with the number i, for $i = 1, 2, 3$. The cards are then removed from the boxes, combined with card with number 4 on it, randomly shuffled, and three of the cards are randomly selected and placed one each into the three boxes; all possibilities of which card is placed in which box are equally likely. Let X denote the number of boxes that get back their original card.

(a) Describe a suitable sample space Ω to describe the experiment. How many elements does Ω have?

Solution: Here is one possibility. We can take

$$\Omega = \{y_1y_2y_3 : y_1y_2y_3 \text{ is a permutation of any three of } 1234\},$$

where y_i represents the number on the card placed back in box i, for $y_i \in \{1, 2, 3, 4\}$. For example, 312 indicates that box 1 gets card numbered 3, box 2 gets number card numbered 1, and box 3 gets card numbered 2. There are $4 \times 3 \times 2 \times 1 = 24$ elements in Ω, i.e., $|\Omega| = 24$.

(b) Find the pmf of X.

Solution: The possible values of X are 0,1, 2, and 3, since there are three boxes in total. There is only one outcome that contributes to $\{X = 3\}$, which is 123, so

$$p_X(3) = \frac{1}{24}.$$

The outcomes in $\{X = 2\}$ correspond to exactly 2-of-3 boxes getting its matching card while the third one has a mismatched card. There are $\binom{3}{2} = 3$ ways of selecting the two boxes with matched cards with the remaining third box containing card numbered 4. Therefore, there are a total of 3 outcomes in $\{X = 2\}$, so

$$p_X(2) = \frac{3}{24}.$$

The outcomes in $\{X = 1\}$ correspond to only one box getting back its number. For each such box i, there three outcomes such that the other two boxes do not have their numbers, e.g., 132, 134, 142 has only box 1 with a matching card. Therefore, there are a total of 9 outcomes in $\{X = 1\}$, so

$$p_X(1) = \frac{9}{24}.$$

Now we can use the fact that the elements of the pmf sum up to one to conclude that

$$p_X(0) = 1 - \frac{1}{24} - \frac{3}{24} - \frac{9}{24} = \frac{11}{24}.$$

(c) Find $E[X]$.
Solution: $E[X] = p_X(1) + 2p_X(2) + 3p_X(3) = 9/24 + 6/24 + 3/24 = 18/24 = 3/4$

(d) Find $\mathrm{Var}(X)$.
Solution: We first find

$$E[X^2] = 1^2 p_X(1) + 2^2 p_X(2) + 3^2 p_X(3) = 9/24 + 12/24 + 9/24 = 30/24$$

and then

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = 30/24 - 9/16 = 33/48$$

5. **[Illini T-Shirts]**
A bag contains 4 orange and 6 blue t-shirts. Two t-shirts are chosen from the bag at random. Let X denote the number of blue t-shirts chosen.

(a) Find the pmf of X.
Solution: The possible values of X are 0,1, or 2. The event $\{X = 0\}$ corresponds to both t-shirts being orange. Therefore

$$p_X(0) = \frac{\binom{4}{2}}{\binom{10}{2}} = \frac{4 \cdot 3}{10 \cdot 9} = \frac{12}{90} = \frac{2}{15}.$$

The event $\{X = 2\}$ corresponds to both t-shirts being blue. Therefore

$$p_X(2) = \frac{\binom{6}{2}}{\binom{10}{2}} = \frac{6 \cdot 5}{10 \cdot 9} = \frac{30}{90} = \frac{5}{15} = \frac{1}{3}.$$

And

$$p_X(1) = 1 - p_X(0) - p_X(2) = \frac{8}{15}.$$

Alternatively, the event $\{X = 1\}$ corresponds to one of the t-shirts being orange and the other blue. Therefore

$$p_X(1) = \frac{\binom{4}{1}\binom{6}{1}}{\binom{10}{2}} = \frac{4 \cdot 6 \cdot 2}{10 \cdot 9} = \frac{48}{90} = \frac{8}{15}.$$

(b) Find $E[(X+1)(X+2)]$.

Solution: By LOTUS,

$$E[(X+1)(X+2)] = 2p_X(0) + 6p_X(1) + 12p_X(2) = \frac{4+48+60}{15} = \frac{112}{15}.$$

6. **[Expected Value]**

If $E[(X+2)^2] = 4$ and $\text{Var}(X) = 1$, find $E[X^2]$ and $E[X]$.

Solution: From linearity of expectation, we get:

$$E[(X+2)^2] = E[X^2] + 4E[X] + 4 = 4 \implies E[X] = -\frac{E[X^2]}{4}$$

From the definition of variance, we get

$$\text{Var}(X) = E[X^2] - (E[X])^2 = 1 \implies E[X] = \pm\sqrt{E[X^2] - 1}$$

Equating the two expressions for $E[X]$, we get the following quadratic equation in $E[X^2]$

$$(E[X^2])^2 - 16E[X^2] + 16 = 0 \implies E[X^2] = 8 \pm 4\sqrt{3} => E[X] = -2 \mp \sqrt{3}$$