University of Illinois **Fall 2025**

ECE 313: Problem Set 1: Problems and Solutions

Due: Friday, September 12 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Chapter 1

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. **[Defining a set of outcomes I]**
   Ten balls, numbered one through ten, are initially in a bag. Three balls are drawn out, one at a time, without replacement.

   (a) Define a sample space Ω describing the possible outcomes of this experiment. To be definite, suppose the order the three balls are drawn out is *unimportant*. Explain how the elements of your set correspond to outcomes of the experiment.
   Solution: A natural choice is $\Omega = \{\{b_1, b_2, b_3\} : 1 \leq b_i \leq 10, b_1, b_2, b_3 \text{ distinct}\}$, where for a given outcome $\{b_1, b_2, b_3\}$, b_i denotes the number on the i^{th} ball drawn from the bag.

   (b) What is the cardinality of Ω?
   Solution: $(10)(9)(8)/(6) = 120$, because there are 10 possible choices for b_1, and given b_1 there are 9 possible choices for b_2, and given b_1 and b_2, there are 8 possible choices for b_3, thus yielding $(10)(9)(8) = 720$. Finally, notice that for a given choice of $\{b_1, b_2, b_3\}$, there are $3! = 6$ permutations that correspond to the same outcome, so we need to divide 720 by 6 which gives the desired result.

2. **[Defining a set of outcomes II]**
   A random experiment consists of selecting two balls in succession from an urn containing two blue balls and one red ball. Assume that each ball from the urn is equally likely to be chosen.

   (a) Suppose that the balls are not replaceable, i.e., the chosen ball in the first selection is removed from the urn. What is the sample space Ω for this experiment?
   Solution: Let (B, R) denote the outcome that the first ball selected is blue and the second is red, and similarly for (B, B), (R, B) and (R, R). With such notation,

$$\Omega = \{(B, B),\ (B, R),\ (R, B)\}$$

Another sample space can be obtained by attaching a subscript to identically colored balls. In this case, the sample space is given by:

$$\Omega = \{(B_1, B_2), (B_1, R), (B_2, B_1), (B_2, R), (R, B_1), (R, B_2)\}$$

In this case, since both sample spaces comprise equally likely outcomes, one can confirm that the probability of obtaining a two blue balls is 1/3 in either case. Note that the second (larger) sample space will always result in equally likely outcomes.

(b) Suppose now that the balls are replaceable, i.e., the chosen ball in the first selection is immediately put back into the urn. What is the sample space Ω for this experiment?

Solution: With the same notation as in (a), we have either

$$\Omega = \{(B, B),\ (B, R),\ (R, B),\ (R, R)\}$$

or

$$\Omega = \{(X, Y)|X, Y \in \{B_1, B_2, R\}\}$$

(c) Considering both of these experiments, does the outcome of the first draw affect the outcome of the second draw? Please briefly justify your answer for both cases.

Solution: If the balls are not replaceable, the first draw will affect the second draw, as there must be a change in the balls available for the second draw. For example, if the first ball selected is red, the second must be blue.

If the balls are replaceable, the first draw will not affect the second draw, as the balls available for the second draw is the same with the first draw.

3. **[Using set theory to calculate probabilities of events]**
Suppose A and B are two events defined on a probability space with $P(A) = 5/6$ and $P(B) = 1/2$.

(a) If $B \subset A$, calculate $P(AB)$.

Solution: If $B \subset A$, then $AB = B$ and $P(AB) = P(B) = 1/2$.

(b) If $A \cup B = \Omega$, calculate $P(AB)$.

Solution: Using relationship:

$$1 = P(\Omega) = P(A \cup B) = P(A) + P(B) - P(AB) = \frac{5}{6} + \frac{1}{2} - P(AB).$$

Hence we get $P(AB) = 4/3 - 1 = 1/3$.

4. **[Possible probability assignments]**
A random experiment has a sample space $\Omega = \{a, b, c, d\}$. Suppose that $P(\{b, c, d\}) = \frac{5}{6}$, $P(\{a, b\}) = \frac{1}{3}$, and $P(\{b, c\}) = \frac{1}{2}$. Use the axioms of probability to find the probabilities of the elementary events ($P(\{a\})$, $P(\{b\})$, $P(\{c\})$, and $P(\{d\})$).

Solution: As a, b, c, d are elements of a sample space, the events $\{a\}, \{b\}, \{c\}, \{d\}$ are mutually exclusive. We also know that $P(\Omega) = P(\{a, b, c, d\}) = 1$. Then

$$P(\{b, c, d\}) = P(\{b\}) + P(\{c\}) + P(\{d\}) = \frac{5}{6} \tag{1}$$

$$P(\{a, b\}) = P(\{a\}) + P(\{b\}) = \frac{1}{3} \tag{2}$$

$$P(\{b, c\}) = P(\{b\}) + P(\{c\}) = \frac{1}{2} \tag{3}$$

$$P(\{a, b, c, d\}) = P(\{a\}) + P(\{b\}) + P(\{c\}) + P(\{d\}) = 1 \tag{4}$$

Solving these equations gives

$$P(\{a\}) = \frac{1}{6}, \quad P(\{b\}) = \frac{1}{6}, \quad P(\{c\}) = \frac{1}{3}, \quad P(\{d\}) = \frac{1}{3}$$

5. **[Displaying outcomes in a two event Karnaugh map]**
Two fair dice are rolled. Let A be the event the sum is 4 and B be the event at least one of the numbers rolled is strictly less than 3.

   (a) Display the outcomes in a Karnaugh map.
   Solution: First, consider event A. Since there are 3 possible (i.e., $4 = 1+3 = 2+2 = 3+1$) out of total 36 ($= 6 \times 6$) possible summations, $P(A) = 3/36 = 1/12$. For event B, one can readily see that $P(B) = 1/3 + 1/3 - 1/3 \times 1/3 = 5/9$. See below for a Karnaugh map, where $(a, b),\ a, b \in \{1, 2, 3, 4, 5, 6\}$ means that a and b are popped up after rolling the first and the second dices, respectively.

	A	A^c
B	$(2,2)$ $(1,3)$ $(3,1)$	$(1,1)$ $(1,2)$ $(1,4)$ $(1,5)$ $(1,6)$ $(2,1)$ $(2,3)$ $(2,4)$ $(2,5)$ $(2,6)$ $(3,2)$ $(4,1)$ $(4,2)$ $(5,1)$ $(5,2)$ $(6,1)$ $(6,2)$
B^c		$(3,3)$ $(3,4)$ $(3,5)$ $(3,6)$ $(4,3)$ $(4,4)$ $(4,5)$ $(4,6)$ $(5,3)$ $(5,4)$ $(5,5)$ $(5,6)$ $(6,3)$ $(6,4)$ $(6,5)$ $(6,6)$

Figure 1: Karnaugh map for Problem 3.

   (b) Determine $P(AB)$.
   Solution: $P(AB) = 3/36 = 1/12$.

6. **[Principles of Counting]**
A restaurant offers 5 entrees: $\{E_1, E_2, E_3, E_4, E_5\}$ and 8 sides: $\{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$. Alice and Bob want to choose 3 different entrees and 4 different sides as their dinner.

   (a) How many different dinners consisting of 3 different entrees and 4 different sides are possible?
   Solution: There are $\binom{5}{3}$ ways to choose entrees and $\binom{8}{4}$ ways to choose sides, so there are
   $$\binom{5}{3}\binom{8}{4} = 10 \times 70 = 700$$
   different dinners.

   (b) Suppose that side from $\{S_1, S_2, S_3, S_4\}$ can only be ordered if E_1 is chosen as one of their entrees, and side from $\{S_5, S_6, S_7, S_8\}$ can be only ordered if both E_2 and E_3 are ordered as two of their entrees. With these constraints, how many different dinners consisting of 3 different entrees and 4 different sides are possible?

Solution: In order to have 4 sides, at least one of (E_1, E_2, E_3) must be ordered. Consider different choices of these entrees:

- If E_1, E_2, E_3 are all selected, then no more entrees are needed, and we can randomly choose 4 sides out of 8.
  \# of ways $= 1 \times \binom{8}{4} = 70$.
- If E_1 is selected but at least one of (E_2, E_3) is not selected, then there are $\binom{4}{2} - 1 = 5$ ways to choose the set of entrees. After that, there is $\binom{4}{4} = 1$ way to choose the sides.
  \# of ways $= 5 \times 1 = 5$.
- If E_1 is not selected and E_2, E_3 are both selected, then there are $\binom{2}{1} = 2$ ways to choose the set of entrees. After that, there is $\binom{4}{4} = 1$ way to choose the sides.
  \# of ways $= 2 \times 1 = 2$.

There is no overlap in the three cases we are considering, and we have considered every possible choice, so totally there are

$$70 + 5 + 2 = 77$$

different dinners.

7. **[Two more poker hands]**
Suppose five cards are drawn from a standard 52 card deck of playing cards, as described in Example 1.4.3, with all possibilities being equally likely.

(a) *FLUSH* is the event that all five cards have the same suit. Find $P(FLUSH)$.
Solution: There are $\binom{13}{5}$ ways to select the numbers for the five cards, then 4 ways to choose the suit. Thus,

$$\begin{aligned} P(FLUSH) &= \frac{\binom{13}{5} \cdot 4}{\binom{52}{5}} \\ &= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 4}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \approx 0.00198 \end{aligned}$$

(b) *SPECIAL* is the event that the five cards are one of A, K, Q, J but not necessarily of the same suit. Find $P(SPECIAL)$.
Solution: Notice that there are 16 cards in total, which are one of A, K, Q, J. Hence,

$$\begin{aligned} P(SPECIAL) &= \frac{\binom{16}{5}}{\binom{52}{5}} \\ &= \frac{1}{595} \approx 0.00168 \end{aligned}$$