**University of Illinois** **Fall 2025**

# ECE 313: Problem Set 0

**Due:** Friday, September 5 at 07:00:00 p.m.
**Reading:**

**Note on reading:** For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand these in; answers to the short answer questions are provided in the appendix of the notes.

**Note on turning in homework:** Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted.

Please write on the top right corner of the first page:
NAME
NETID
SECTION
PROBLEM SET #

Page numbers are encouraged but not required. Five points will be deducted for improper headings. Please assign your uploaded pages to their respective question numbers while submitting your homework on Gradescope. **5 points will be deducted for incorrectly assigned pages.**

1. **[MacLaurin, Taylor Series; L'Hopital's Rule ]**
   Solve the following problems.

   (a) Prove that $1 + x + x^2 + \cdots + x^{n-1} = \frac{1-x^n}{1-x}$ for all $x \neq 1$ and integers $n \geq 1$.

   (b) Assume that $n$ is a positive integer. Show that $1 + x + x^2 + \cdots + x^{n-1} = \lim_{x \to 1} \frac{1-x^n}{1-x}$ when $x = 1$.

   (c) Assume that $|x| < 1$. Find the sum of the series $1 + x + x^2 + \cdots$ without directly using the Geometric sum formula. **Hint:** think about the limit of the finite sum in part (a) when $n \to \infty$.

   (d) Prove that

   $$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

   using L'Hôpital's rule.

   (e) Prove that

   $$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

   using the MacLaurin series for $\sin(x)$.

   (f) Using the expression you found in part (e), find $d(\frac{\sin(x)}{x})/dx$ at $x = 0$.

2. **[The Binomial Theorem]**
   Solve the following problems.

   (a) For positive integers $n$ and $k$, compute the $k$-th derivative of $(1 + x)^n$ using the chain rule. Use these derivatives to find the Taylor expansion of $(x + 1)^n$ around 0. Then, using what you have found, show that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

(b) Now consider the function $g(x) = (1-x)^{-n}$. Is the Taylor expansion of this function around 0 the same as part a? Show work explaining why or why not.

(c) Write down the Taylor expansion around 0 for functions $(1-x)^{-1}$ and $(1-x)^{-2}$.

(d) Write down the Taylor expansion around 0 for function $(1+x)^{\alpha}$, where $\alpha$ is not necessarily an integer.

3. **[Function Extrema]**
Solve the following problems.

(a) Find all maximum and minimum values (if any) of the function $x^{25}(1.00001)^{-x}$ on the interval $(0, \infty)$.

(b) Find all maximum and minimum values (if any) of the function $e^{-|x|}$ on the interval $[-1, 2]$.

(c) Find $k^*$ that maximizes the function:

$$f(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where $n \geq k$ is a non-negative integer.

4. **[Definite Integrals]**
Solve the following definite integrals.

(a)

$$\int_{-2}^{1} |x| dx$$

(b)

$$\int_{0}^{1} x(1-x^2)^{11} dx$$

(c)

$$\int_{0}^{1} x^2 e^{-x} dx$$

(d)

$$\int_{-10}^{10} x^3 e^{\frac{-x^2}{2}}$$

5. **[Derivatives and Integrals]**
Let $\frac{d}{dx} f(x) = g(x)$, $f(x) > 0$, $-\infty < x < \infty$, and let $C$ be an arbitrary constant. Which of the following statements are true for all $x$?

(a)

$$\frac{d}{dx} f(-x) = -g(-x)$$

(b)

$$\frac{d}{dx} f(\frac{x^2}{2}) = xg(\frac{x^2}{2})$$

(c)

$$\frac{d}{dx} e^{f(x^2)} = g(x^2)e^{f(x^2)}$$

(d)

$$\int g(-x)dx = f(-x) + C$$

(e)

$$\int g(\frac{x^2}{2})dx = \frac{f(\frac{x^2}{2})}{x} + C$$

(f)

$$\int \frac{g(x)}{f(x)} = \ln(f(x)) + C$$

6. **[Double Integrals]**
   Evaluate the following two-dimensional integrals over their specified domains.

   (a) Integrate $f(x,y) = \min(x,y)$ over the region $\{(x,y) : 0 \leq x \leq 2, 0 \leq y \leq 1\}$

   (b) Integrate $f(x,y) = \exp(-\frac{1}{2}(x^2+y^2))$ over the region $\{(x,y) : x^2+y^2 > 4\}$ **Hint:** change to polar coordinates ($x^2+y^2 = 4$ denotes a circle). You may have to do a $u$-substitution.