University of Illinois Fall 2025

ECE 313: Hour Exam II

Monday, November 3, 2025
7:00 p.m. — 8:30 p.m.

1. **[25 points]** Let X denote the outcome of rolling a fair die. Define the random variable $Y = 2X + 1$.

   (a) (15 points) Bound $P(Y \geq 10)$ using Markov's inequality.
   Solution:
   $$Y = 2X + 1, \quad \mathbb{E}[X] = 3.5 \implies \mathbb{E}[Y] = 2(3.5) + 1 = 8.$$
   By Markov's inequality:
   $$P(Y \geq 10) \leq \frac{\mathbb{E}[Y]}{10} = \frac{8}{10} = 0.8.$$
   $$\boxed{P(Y \geq 10) \leq 0.8.}$$

   (b) (10 points) Bound $P(Y \geq 12)$ using Chebyshev's inequality.
   Solution:
   $$\mathrm{Var}(X) = \frac{35}{12} \implies \mathrm{Var}(Y) = 4 \cdot \frac{35}{12} = \frac{35}{3}.$$
   $$\mu_Y = 8, \quad P(Y \geq 12) = P(Y - 8 \geq 4) \leq P(|Y - 8| \geq 4).$$
   By Chebyshev:
   $$P(|Y - 8| \geq 4) \leq \frac{\mathrm{Var}(Y)}{4^2} = \frac{35/3}{16} = \frac{35}{48}.$$
   $$\boxed{P(Y \geq 12) \leq \frac{35}{48}.}$$

2. **[25 points]** The ECE 313 instructors in wish to come up with decision rules that allows them to predict whether a student has taken ECE 210 previously or not based on their ECE 313 grade. The following historical data is available:

   - From the students who took ECE 210 previously, 45% earned an A, 45% earned a B, and 10% earned a C in ECE 313.
   - From the students who did not take ECE 210, 7.5% earned an A, 57.5% earned a B, and 35% earned a C in ECE 313.

   Let X denote a student's letter grade in ECE 313, H_1 represent the hypothesis that the "student took ECE 210", and H_0 represent the hypothesis that "the student did not take ECE 210 previously".

   (a) (10 points) Construct the likelihood matrix and find the ML decision rule.

   Solution:

	$X = A$	$X = B$	$X = C$
H_1	<u>0.45</u>	0.45	0.1
H_0	0.075	<u>0.575</u>	<u>0.35</u>

(b) (10 points) Assume 60% of the students took ECE 210 before, find the MAP decision rule using the joint probability matrix.

Solution: We compute $P(X, H_i) = P(X \mid H_i) \cdot P(H_i)$ for each x:

	$X = A$	$X = B$	$X = C$
H_1	$\underline{0.27}$	$\underline{0.27}$	0.06
H_0	0.03	0.23	$\underline{0.14}$

(c) (5 points) Find $p_{\text{false-alarm}}$ and p_{miss} for the MAP rule.

Solution:

$$p_{\text{false-alarm}} = P(\text{Claim } H_1 | H_0) = \frac{0.03 + 0.23}{0.4} = 0.65,$$

and

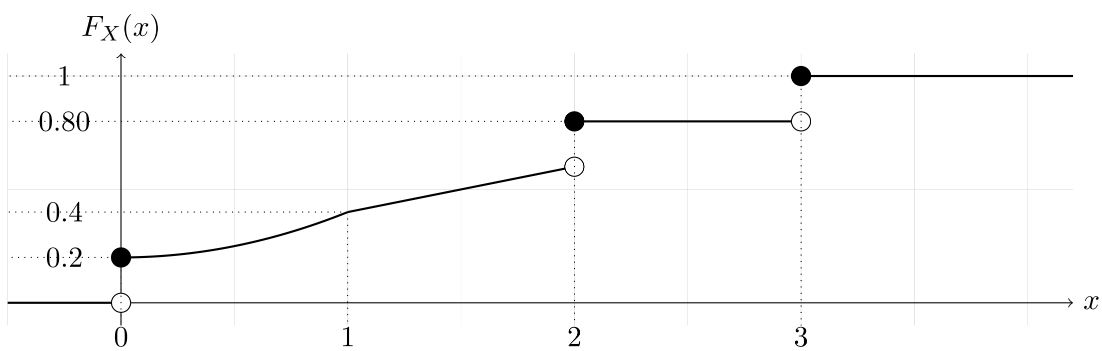
$$p_{\text{miss}} = P(\text{Claim } H_0 | H_1) = \frac{0.06}{0.6} = 0.1$$

3. **[25 points]** The random variable X has the following cumulative distribution function (CDF):

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.2 + 0.2x^2, & 0 \le x \le 1, \\ 0.4 + 0.2(x-1), & 1 < x < 2, \\ 0.80, & 2 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

(a) (10 points) Sketch a plot of $F_X(x)$.

Solution:



(b) (15 points) Compute the probabilities of the following events: $P\{X = 0\}$, $P\{0.5 < X < 1.5\}$, $P\{X = 2\}$, $P\{X = 0.7\}$, and $P\{0 < X \le 2\}$.

Solution:

$$P\{X = 0\} = F_X(0) - F_X(0^-) = 0.2 - 0 = 0.2$$
$$P\{0.5 < X < 1.5\} = F_X(1.5^-) - F_X(0.5) = F_X(1.5) - F_X(0.5) = 0.5 - 0.25 = 0.25$$
$$P\{X = 2\} = F_X(2) - F_X(2^-) = 0.8 - 0.6 = 0.2$$
$$P\{X = 0.7\} = F_X(0.7) - F_X(0.7^-) = F_X(0.7) - F_X(0.7) = 0$$
$$P\{0 < X \le 2\} = F_X(2) - F_X(0) = 0.8 - 0.2 = 0.6$$

4. **[25 points]** Answer the following:

(a) (8 points) If $X \sim \mathcal{N}(1, 4)$, find the probability $P\{2|X-1| \geq 4\}$. Express your answer in terms of the Q function.

Solution:

$$
\begin{aligned}
P\{2|X-1| \geq 4\} &= P\{|X-1| \geq 2\} = P\{\{X-1 \geq 2\} \cup \{X-1 \leq -2\}\} \\
&= P\{X-1 \geq 2\} + P\{X-1 \leq -2\} \\
&= P\left\{\frac{X-1}{2} \geq 1\right\} + P\left\{\frac{X-1}{2} \leq -1\right\} = Q(1) + \Phi(-1) \\
&= 2Q(1)
\end{aligned}
$$

(b) (9 points) A fisherman catches fish in a river according to a Poisson process with rate equal to 2 fishes per half hour. Find the probability that he will catch exactly one fish between 8 AM and 9 AM, and a total of exactly 3 fish by 9:30 AM? Leave your answers in terms of powers of e and its multiples.

Solution: Let X_2 denote the number of fish he catches between 8 AM and 9 AM, and X_3 denote the number of fish that he catches between 9 AM and 9:30 AM. Then, $X_2 + X_3 = 3$ where $X_2 \sim \text{Poi}(4)$ and $X_3 \sim \text{Poi}(2)$. By the independent increment property of the Poisson process, X_2 and X_3 are independent. Therefore, the desired probability is given by:

$$
P\{X_2 = 1, X_3 = 2\} = P\{X_2 = 1\}P\{X_3 = 2\} = 4e^{-4} \times \frac{2^2 e^{-2}}{2!} = 8e^{-6} \approx 19.83 \times 10^{-3}.
$$

(c) (8 points) If $Y = aX + b$ such that $E[Y] = 11$ and $\text{Var}(Y) = 3$ when $X \sim \text{Unif}[2, 4]$, determine the values of a and b and the pdf of Y, $f_Y(v)$.

Solution: Since $X \sim \text{Unif}[2, 4]$,

$$
E[X] = \frac{2+4}{2} = 3; \quad \text{Var}(X) = \frac{(4-2)^2}{12} = \frac{1}{3}
$$

From linearity of expectation,

$$
E[Y] = aE[X] + b = 3a + b; \quad \text{Var}(Y) = a^2 \text{Var}(X) = \frac{a^2}{3} \implies (a, b) = (3, 2) \text{ or } (-3, 20)
$$

Since Y is a linearly scaled version of X, it is also a uniformly distributed random variable. Since the support of $Y \in [8, 14]$ and hence $Y \sim \text{Unif}[8, 14]$, i.e., $f_Y(v) = \frac{1}{6}$ for $v \in [8, 14]$ and $f_Y(v) = 0$ otherwise.