University of Illinois Fall 2025

ECE 313: Hour Exam I

Monday, October 6, 2025
7:00 p.m. — 8:30 p.m.

1. **[25 points]** Let A, B, C be 3 events such that $P(A) = 3/4$, $P(AB) = 1/2$ and $C \subset B$.

   (a) (15 points) If $P(A^cB^c) = 1/8$, find $P(B)$.
   Solution: We are given $P(A) = \frac{3}{4}$, hence $P(A^c) = \frac{1}{4}$.
   By the law of total probability with respect to A:
   $$P(B) = P(AB) + P(A^cB).$$
   Now
   $$P(A^cB) = P(A^c) - P(A^cB^c) = \tfrac{1}{4} - \tfrac{1}{8} = \tfrac{1}{8}.$$
   Therefore,
   $$P(B) = \tfrac{1}{2} + \tfrac{1}{8} = \tfrac{5}{8}.$$

   (b) (10 points) **True or False**: $P(AC) <= 2/3$. Justify your answer.
   Solution: Since $C \subset B$, we have $AC \subseteq AB$. Therefore,
   $$P(AC) \le P(AB) = \tfrac{1}{2}.$$
   Clearly,
   $$\tfrac{1}{2} < \tfrac{2}{3},$$
   so the statement is **True**.

2. **[25 points]** A discrete random variable X has the following distribution
$$P\{X = k\} = \frac{n}{k},\ \text{for } k \in \{1, 2, 3\}, \quad P\{X = k\} = \frac{m}{2k},\ \text{for } k = 6,$$
and $P\{X = k\} = 0$ for all remaining values of k. Here, n and m are unknown parameters of the distribution.

   (a) (15 points) Assume that the parameters n and m are such that $E[X] = 3$. Solve for n and m, and find $\mathrm{Var}(X)$.
   Solution: We can find a formula for expected value.
   $$\begin{aligned} E[X] &= 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) + 6 \cdot P(X = 6) && (1)\\ &= 3n + \frac{m}{2} = 3 && (2) \end{aligned}$$
   We also know that, to be a valid pmf, all probabilities must sum to one.
   $$\begin{aligned} n + \frac{n}{2} + \frac{n}{3} + \frac{m}{12} &= \frac{22n + m}{12} && (3)\\ &= 1 && (4) \end{aligned}$$
   Solving this linear system of equations gives us:
   $$n = \frac{3}{8}$$

$$m = \frac{15}{4}$$

Therefore,

$$E[X^2] = 1 \cdot P(X = 1) + 4 \cdot P(X = 2) + 9 \cdot P(X = 3) + 36 \cdot P(X = 6) \tag{5}$$
$$= 6n + 3m = \frac{27}{2} \tag{6}$$

And

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{27}{2} - (3)^2 = \frac{9}{2}$$

(b) (5 points) Using the values of n and m found in part (a), find $E[3X^2 - 4]$.
Solution: Using linearity of Expectation,

$$E[3X^2 - 4] = 3E[X^2] - 4 = 3 \cdot \frac{27}{2} - 4 = \frac{73}{2} \tag{7}$$

(c) (5 points) Assume you know nothing about $E[X]$, but you know that $m = 1$. Let Y be a geometric RV with parameter p. Find the value of p such that $P\{X = 2\} = P\{Y = 2\}$.
Solution: Since the pmf of X must sum to 1, we find that $n = \frac{1}{2}$. Therefore,

$$P(X = 2) = \frac{1}{4} = P(Y = 2) = p(1 - p) \implies p = \frac{1}{2}$$

3. **[25 points] V1 Spam Filter Using Bayes' Theorem**

A spam filter classifies emails as spam or not spam. Based on historical data:

- 40% of all emails are spam.
- 24% of all emails are spam and contain the word "lottery."
- 3% of all emails are not spam and contain the word "lottery."
- 10% of all emails are spam and contain the word "urgent."
- 6% of all emails are not spam and contain the word "urgent."
- 5% of all emails are spam and contain both "lottery" and "urgent."
- 1% of all emails are not spam and contain both "lottery" and "urgent."

Define events: S: Email is spam, S^c: Email is not spam, L: Email contains "lottery", and U: Email contains "urgent".

(a) (13 points) An email contains both "lottery" and "urgent." What is the probability it is spam, i.e., $P(S \mid L \cap U)$?
Solution: We want to compute:

$$P(S \mid L \cap U) = \frac{P(L \cap U \mid S) \cdot P(S)}{P(L \cap U)} \tag{Bayes' theorem}$$

Numerator:

$$P(L \cap U \mid S) \cdot P(S) = P(S \cap L \cap U) = 0.05$$

Denominator:

$$P(L \cap U) = P(S \cap L \cap U) + P(S^c \cap L \cap U) = 0.05 + 0.01 = 0.06$$

Therefore:

$$P(S \mid L \cap U) = \frac{0.05}{0.06} = \frac{5}{6}$$

(b) (12 points) An email contains neither "lottery" nor "urgent." What is the probability it is spam, i.e., $P(S \mid L^c \cap U^c)$?
Solution: We want to compute:

$$P(S \mid L^c \cap U^c) = \frac{P(L^c \cap U^c \mid S) \cdot P(S)}{P(L^c \cap U^c)} \qquad \text{(Bayes' theorem)}$$

Numerator:

$$\begin{aligned} P(L^c \cap U^c \mid S) \cdot P(S) &= P(S \cap L^c \cap U^c) \\ &= P(S) - P(S \cap L) - P(S \cap U) + P(S \cap L \cap U) \\ &= 0.40 - 0.24 - 0.10 + 0.05 = 0.11 \end{aligned}$$

Denominator:

$$\begin{aligned} P(L^c \cap U^c) &= P(S \cap L^c \cap U^c) + P(S^c \cap L^c \cap U^c) \\ &= 0.11 + [P(S^c) - P(S^c \cap L) - P(S^c \cap U) + P(S^c \cap L \cap U)] \\ &= 0.11 + (0.60 - 0.03 - 0.06 + 0.01) = 0.11 + 0.52 = 0.63 \end{aligned}$$

Therefore:

$$P(S \mid L^c \cap U^c) = \frac{0.11}{0.63} = \frac{11}{63}$$

4. **[25 points]** Consider a lottery of the following rule: The guest pick 3 different numbers from 1 to 10 for a lottery ticket. At the end of each day, the host will draw 3 winning numbers at random. There are two prizes:
(a) Grand Prize: All 3 picked numbers match 3 winning numbers.
(b) Small Prize: Exactly 2 picked numbers match any 2 out of the 3 winning numbers.

(a) (5 points) If the grand prize is \$100 and the small prize is \$5. What is the expected prize amount of each ticket?
Solution: Let X denotes the event of grand prize and Y denote the event of small prize. From counting, we have

$$\begin{aligned} P(X) &= \frac{||X||}{||\Omega||} = \frac{\binom{3}{3}}{\binom{10}{3}} = \frac{1}{120} \\ P(Y) &= \frac{\binom{3}{2} \times 7}{120} = \frac{7}{40} \end{aligned}$$

The expected prize amount is then $100 \times P(X) + 5 \times P(Y) = \frac{5}{6} + \frac{7}{8} = \frac{41}{24}$.

(b) (10 points) Let X denote the number of grand prizes won with 10 tickets, find $E[X]$, $\mathrm{Var}(X)$, and pmf $p_X(k)$ in terms of k.
Solution: $X \sim \mathrm{Bi}(n = 10, p = \frac{1}{120})$. We have

$$\begin{aligned} E[X] &= np = \frac{1}{12} \\ Var(X) &= np(1-p) = \frac{119}{1440}. \\ p_X(k) &= \binom{10}{k} (\frac{1}{120})^k (\frac{119}{120})^{10-k} \end{aligned}$$

(c) (10 points) Assume Alice buys a lottery ticket everyday. Let Z be the number of days until Alice first wins any prize (Grand or Small). Find $E[Z]$ and pmf $p_Z(k)$ in terms of k

Solution: Note that the grand prize and the small prize are mutually exclusive. As a result, the probability of wining any of them is sum of two probability $\frac{1}{120} + \frac{7}{40} = \frac{11}{60}$. $Z = Geo(p = \frac{11}{60})$. We have

$$E[Z] = \frac{1}{p} = \frac{60}{11}$$
$$p_Z(k) = (\frac{49}{60})^{k-1}(\frac{11}{60})$$