University of Illinois Fall 2025

ECE 313: Hour Exam II (Conflict)

Tuesday, November 4, 2025
8:00 a.m. — 9:30 a.m.

1. **[25 points]** Let X denote the outcome of rolling a fair die. Define the random variable $Y = 4X + 2$.

   (a) (15 points) Bound $P(Y \geq 24)$ using Markov's inequality.
   Solution:
   $$Y = 4X + 2, \quad \mathbb{E}[X] = 3.5 \implies \mathbb{E}[Y] = 4(3.5) + 2 = 16.$$
   By Markov's inequality:
   $$P(Y \geq 24) \leq \frac{\mathbb{E}[Y]}{24} = \frac{16}{24} = \frac{2}{3}.$$
   $$\boxed{P(Y \geq 24) \leq \frac{2}{3}.}$$

   (b) (10 points) Bound $P(Y \geq 24)$ using Chebyshev's inequality.
   Solution:
   $$\mathrm{Var}(X) = \frac{35}{12} \implies \mathrm{Var}(Y) = 16 \cdot \frac{35}{12} = \frac{140}{3}.$$
   $$\mu_Y = 16, \quad P(Y \geq 24) = P(Y - 16 \geq 8) \leq P(|Y - 16| \geq 8).$$
   By Chebyshev:
   $$P(|Y - 16| \geq 8) \leq \frac{\mathrm{Var}(Y)}{8^2} = \frac{(140/3)}{64} = \frac{35}{48}.$$
   $$\boxed{P(Y \geq 24) \leq \frac{35}{48}.}$$

2. **[25 points]** The ECE 313 instructors in wish to come up with decision rules that allows them to predict whether a student has taken ECE 210 previously or not based on their ECE 313 grade. The following historical data is available:

   - From the students who took ECE 210 previously, 40% earned an A, 40% earned a B, and 20% earned a C in ECE 313.
   - From the students who did not take ECE 210, 10% earned an A, 50% earned a B, and 40% earned a C in ECE 313.

   Let X denote a student's letter grade in ECE 313, H_1 represent the hypothesis that the "student took ECE 210", and H_0 represent the hypothesis that "the student did not take ECE 210 previously".

   (a) (10 points) Construct the likelihood matrix and find the ML decision rule.

   Solution:

	$X = A$	$X = B$	$X = C$
H_1	$\underline{0.4}$	0.4	0.2
H_0	0.1	$\underline{0.5}$	$\underline{0.4}$

(b) (10 points) Assume 60% of the students took ECE 210 before, find the MAP decision rule using the joint probability matrix.

Solution: We compute $P(X, H_i) = P(X \mid H_i) \cdot P(H_i)$ for each x:

	$X = A$	$X = B$	$X = C$
H_1	$\underline{0.24}$	$\underline{0.24}$	0.12
H_0	0.04	0.2	$\underline{0.16}$

(c) (5 points) Find $p_{\text{false-alarm}}$ and p_{miss} for the MAP rule.

Solution:

$$p_{\text{false-alarm}} = P(\text{Claim } H_1 | H_0) = \frac{0.04 + 0.2}{0.4} = 0.6$$

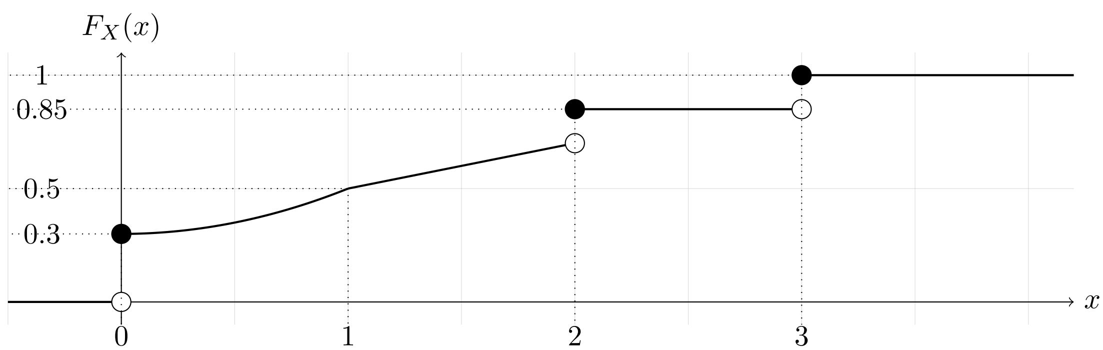
and

$$p_{\text{miss}} = P(\text{Claim } H_0 | H_1) = \frac{0.12}{0.6} = 0.2$$

3. **[25 points]** The random variable X has the following cumulative distribution function (CDF):

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 0.3 + 0.2x^2, & 0 \le x \le 1, \\ 0.5 + 0.2(x-1), & 1 < x < 2, \\ 0.85, & 2 \le x < 3, \\ 1, & x \ge 3. \end{cases}$$

(a) (10 points) Sketch a plot of $F_X(x)$. **Solution:**



(b) (15 points) Compute the probabilities of the following events: $P\{X = 0\}$, $P\{0.5 < X < 1.5\}$, $P\{X = 2\}$, $P\{X = 0.7\}$, and $P\{0 < X \le 2\}$.

Solution:

$$P\{X = 0\} = F_X(0) - F_X(0^-) = 0.3 - 0 = 0.3$$
$$P\{0.5 < X < 1.5\} = F_X(1.5^-) - F_X(0.5) = F_X(1.5) - F_X(0.5) = 0.6 - 0.35 = 0.25$$
$$P\{X = 2\} = F_X(2) - F_X(2^-) = 0.85 - 0.7 = 0.15$$
$$P\{X = 0.7\} = F_X(0.7) - F_X(0.7^-) = F_X(0.7) - F_X(0.7) = 0$$
$$P\{0 < X \le 2\} = F_X(2) - F_X(0) = 0.85 - 0.3 = 0.55$$

4. **[25 points]** Answer the following:

(a) (8 points) If $X \sim \mathcal{N}(1, 4)$, find the probability $P\{(X-1)^2 \geq 4\}$. Express your answer in terms of the Q function.

Solution:

$$\begin{aligned} P\{(X-1)^2 \geq 4\} &= P\{\{X-1 \geq 2\} \cup \{X-1 \leq -2\}\} \\ &= P\{X-1 \geq 2\} + P\{X-1 \leq -2\} \\ &= P\left\{\frac{X-1}{2} \geq 1\right\} + P\left\{\frac{X-1}{2} \leq -1\right\} = Q(1) + \Phi(-1) \\ &= 2Q(1) \end{aligned}$$

(b) (9 points) A fisherman catches fish in a river according to a Poisson process with rate equal to 2 fishes per half hour. He starts fishing at 8 AM. Find the probability that he catches his third fish before 9 AM. Leave your answers in terms of powers of e and its multiples.

Solution: Let X denote the number of fish he catches between 8 AM and 9 AM. Then, $X \sim \text{Poi}(4)$. Therefore, the desired probability is given by:

$$\begin{aligned} P\{X \geq 3\} = 1 - P\{X \leq 2\} &= 1 - (P\{X=0\} + P\{X=1\} + P\{X=2\}) \\ &= 1 - e^{-4}(1 + 4 + \frac{4^2}{2}) = 1 - 13e^{-4} \approx 0.76. \end{aligned}$$

(c) (8 points) If $Y = aX + b$ such that $E[Y] = 11$ and $\text{Var}(Y) = 3$ when $X \sim \text{Unif}[2, 4]$, determine the values of a and b and the pdf of Y, $f_Y(v)$.

Solution: Since $X \sim \text{Unif}[2, 4]$,

$$E[X] = \frac{2+4}{2} = 3; \quad \text{Var}(X) = \frac{(4-2)^2}{12} = \frac{1}{3}$$

From linearity of expectation,

$$E[Y] = aE[X] + b = 3a + b; \quad \text{Var}(Y) = a^2\text{Var}(X) = \frac{a^2}{3} \implies a = 3; \quad b = 2 \quad \text{or} \quad a = -3; b = 20$$

Since Y is a linearly scaled version of X, it is also a uniformly distributed random variable. In both scenarios, the support of $Y \in [8, 14]$ and hence $Y \sim \text{Unif}[8, 14]$, i.e., $f_Y(v) = \frac{1}{6}$ for $v \in [8, 14]$ and $f_Y(v) = 0$ otherwise.