University of Illinois Fall 2025

ECE 313: Hour Exam I (Conflict)

Tuesday, October 7, 2025
8:00 a.m. — 9:30 a.m.

1. **[25 points]** Let A, B, C be 3 events such that $P(A) = 1/2$, $P(A \cup B) = 3/4$ and $C \subset B$.

   (a) (15 points) If $P(AB^c) = 1/4$, find $P(B)$.
   Solution:
   $$P(A \cup B) = P(A) + P(B) - P(AB).$$
   We are given:
   $$P(AB^c) = \tfrac{1}{4}.$$
   Thus,
   $$P(A) = P(AB) + P(AB^c) \;\Rightarrow\; P(AB) = P(A) - P(AB^c) = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}.$$
   Substitute into the union formula:
   $$\tfrac{3}{4} = \tfrac{1}{2} + P(B) - \tfrac{1}{4} \;\Rightarrow\; P(B) = \tfrac{1}{2}.$$

   (b) (10 points) **True or False**: $P(A \cup C) > 4/5$. Justify your answer.
   Solution: Since $C \subset B$, we have $A \cup C \subseteq A \cup B$. Therefore,
   $$P(A \cup C) \leq P(A \cup B) = \tfrac{3}{4}.$$
   Because $\tfrac{3}{4} < \tfrac{4}{5}$, the statement
   $$P(A \cup C) > \tfrac{4}{5}$$
   is **False**.

2. **[25 points]** A discrete random variable X has the following distribution:
   $$P\{X = k\} = \frac{n}{k}, \text{ for } k \in \{1, 2, 3\}, \quad P\{X = k\} = \frac{m}{2k}, \text{ for } k = 6,$$
   and $P\{X = k\} = 0$ for all remaining values of k. Here, n and m are unknown parameters of the distribution.

   (a) (15 points) Assume that the parameters n and m are such that $E[X] = 3$. Solve for n and m, and find $\mathrm{Var}(X)$.
   Solution: We can find a formula for expected value.
   $$E[X] = 1 \cdot P(X = 1) + 2 \cdot P(X = 2) + 3 \cdot P(X = 3) + 6 \cdot P(X = 6) \tag{1}$$
   $$= 3n + \frac{m}{2} = 3 \tag{2}$$
   We also know that, to be a valid pmf, all probabilities must sum to one.
   $$n + \frac{n}{2} + \frac{n}{3} + \frac{m}{12} = \frac{22n + m}{12} \tag{3}$$
   $$= 1 \tag{4}$$

Solving this linear system of equations gives us:

$$n = \frac{3}{8}$$

$$m = \frac{15}{4}$$

Therefore,

$$\begin{aligned} E[X^2] &= 1 \cdot P(X = 1) + 4 \cdot P(X = 2) + 9 \cdot P(X = 3) + 36 \cdot P(X = 6) && (5) \\ &= 6n + 3m = \frac{27}{2} && (6) \end{aligned}$$

And

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{27}{2} - (3)^2 = \frac{9}{2}$$

(b) (10 points) Again assume you know nothing about $E[X]$, but you know $m = 0$. Let Z be a Bernoulli RV whose conditional pmf given X equals:

$$P\{Z = 0 | X = k\} = \frac{k}{6}.$$

Find $E[Z]$.

Solution: Since $m = 0$, and the support for X is $\{1, 2, 3\}$, we notice that:

$$\sum_{k=1}^{3} \frac{n}{k} = n + \frac{n}{2} + \frac{n}{3} = 1 \implies n = \frac{6}{11}$$

Thus, we have the pmf of X, $p_X(k)$ $(k = 1, 2, 3)$.
Next, using the Law of Total Probability, we get:

$$P\{Z = 0\} = \sum_{x=1}^{3} P\{Z = 0 | X = k\} \cdot P\{X = k\} = \frac{1}{6} \cdot \frac{6}{11} + \frac{2}{6} \cdot \frac{3}{11} + \frac{3}{6} \cdot \frac{2}{11} = \frac{3}{11}$$

Therefore, $P\{Z = 1\} = 1 - \frac{3}{11} = \frac{8}{11}$, which implies $E[Z] = 1 \cdot P\{Z = 1\} = \frac{8}{11}$.

3. **[25 points] V2 Spam Filter Using Bayes' Theorem**

A spam filter classifies emails as spam or not spam. Based on historical data:

- 40% of all emails are spam.
- 24% of all emails are spam and contain the word "lottery."
- 3% of all emails are not spam and contain the word "lottery."
- 10% of all emails are spam and contain the word "urgent."
- 6% of all emails are not spam and contain the word "urgent."
- 5% of all emails are spam and contain both "lottery" and "urgent."
- 1% of all emails are not spam and contain both "lottery" and "urgent."

Define events: S: Email is spam, S^c: Email is not spam, L: Email contains "lottery", and U: Email contains "urgent".

(a) (12.5 points) An email contains "lottery" but not "urgent." What is the probability it is spam, i.e., $P(S \mid L \cap U^c)$?
Solution: We want to compute:

$$P(S \mid L \cap U^c) = \frac{P(L \cap U^c \mid S) \cdot P(S)}{P(L \cap U^c)} \qquad \text{(Bayes' theorem)}$$

Numerator:

$$\begin{aligned} P(L \cap U^c \mid S) \cdot P(S) &= P(S \cap L \cap U^c) \\ &= P(S \cap L) - P(S \cap L \cap U) \qquad \text{(Total Probability Formula)} \\ &= 0.24 - 0.05 = 0.19 \end{aligned}$$

Denominator:

$$\begin{aligned} P(L \cap U^c) &= P(S \cap L \cap U^c) + P(S^c \cap L \cap U^c) \\ &= [P(S \cap L) - P(S \cap L \cap U)] + [P(S^c \cap L) - P(S^c \cap L \cap U)] \\ &\qquad \text{(Total Probability Formula)} \\ &= (0.24 - 0.05) + (0.03 - 0.01) = 0.19 + 0.02 = 0.21 \end{aligned}$$

Therefore:

$$P(S \mid L \cap U^c) = \frac{0.19}{0.21} = \frac{19}{21}$$

(b) (12.5 points) An email contains "urgent" but not "lottery." What is the probability it is spam, i.e., $P(S \mid L^c \cap U)$?
Solution: We want to compute:

$$P(S \mid L^c \cap U) = \frac{P(L^c \cap U \mid S) \cdot P(S)}{P(L^c \cap U)} \qquad \text{(Bayes' theorem)}$$

Numerator:

$$\begin{aligned} P(L^c \cap U \mid S) \cdot P(S) &= P(S \cap L^c \cap U) \\ &= P(S \cap U) - P(S \cap L \cap U) \qquad \text{(Total Probability Formula)} \\ &= 0.10 - 0.05 = 0.05 \end{aligned}$$

Denominator:

$$\begin{aligned} P(L^c \cap U) &= P(S \cap L^c \cap U) + P(S^c \cap L^c \cap U) \\ &= [P(S \cap U) - P(S \cap L \cap U)] + [P(S^c \cap U) - P(S^c \cap L \cap U)] \\ &\qquad \text{(Total Probability Formula)} \\ &= (0.10 - 0.05) + (0.06 - 0.01) = 0.05 + 0.05 = 0.10 \end{aligned}$$

Therefore:

$$P(S \mid L^c \cap U) = \frac{0.05}{0.10} = \frac{1}{2}$$

4. **[25 points]** Consider a lottery of the following rule: The guest pick 3 different numbers from 1 to 10 for a lottery ticket. At the end of each day, the host will draw 3 winning numbers at random. There are two prizes:
(a) Grand Prize: All 3 picked numbers match 3 winning numbers.
(b) Small Prize: Exactly 2 picked numbers match any 2 out of the 3 winning numbers.

(a) (5 points) If the grand prize is \$100 and the small prize is \$5. What is the expected prize amount of each ticket?

Solution: Let X denotes the event of grand prize and Y denote the event of small prize. From counting, we have

$$P(X) = \frac{||X||}{||\Omega||} = \frac{\binom{3}{3}}{\binom{10}{3}} = \frac{1}{120}$$
$$P(Y) = \frac{\binom{3}{2} \times 7}{120} = \frac{7}{40}$$

The expected prize amount is then $100 \times P(X) + 5 \times P(Y) = \frac{5}{6} + \frac{7}{8} = \frac{41}{24}$.

(b) (10 points) Assume Alice buys a lottery ticket everyday. Let Z be the number of days until Alice first wins any prize (Grand or Small). Find $E[Z]$ and pmf $p_Z(k)$ in terms of k.

Solution: Note that the grand prize and the small prize are mutually exclusive. As a result, the probability of winning any of them is sum of two probability $\frac{1}{120} + \frac{7}{40} = \frac{11}{60}$. $Z = Geo(p = \frac{11}{60})$. We have

$$E[Z] = \frac{1}{p} = \frac{60}{11}$$
$$p_Z(k) = (\frac{49}{60})^{k-1}(\frac{11}{60})$$

(c) (10 points) Let V be the number of days until Alice first wins 2 grand prizes. Find $E[V]$ and pmf $p_V(k)$ in terms of k.

Solution: V follows a convention negative binomial distribution $V \sim NB(r = 2, p = \frac{1}{120})$. We have

$$E[V] = r \times \frac{1}{p} = 240$$
$$p_V(k) = \binom{k-1}{r-1}(1-p)^{k-r}p^r = (k-1)(\frac{119}{120})^{k-2}(\frac{1}{120})^2$$