ECE 313: Lecture 37
Minimum MSE (mean square error) estimators: constant, unconstrained, and linear

Prob / Application :

$$X \longrightarrow Y$$

estimate $\hat{Y}$

Mean square error

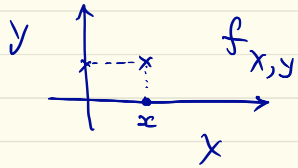
$$MSE = E\left[(\hat{Y}-Y)^2\right]$$

Ex : $X = Y + N$

observed signal that is noisy — X

unknown true signal — Y

noise — N



g estimator : $\hat{Y} = g(x)$

$$MSE_g = E\left[(g(X) - Y)^2\right] = \iint (g(x)-y)^2 f_{X,Y}\, dx\, dy$$

Three estimators (algorithms)

1. Ignore X, Constant : $\hat{Y} = g(X) = \delta^*$

2. Linear : $\hat{Y} = g(X) = a^* X + b^*$

3. Unconstrained : $\hat{Y} = g(X)$

no restriction on g

① Constant estimator

$$Y \rightarrow f_Y$$

$$\hat{Y} = g(X) = \delta$$

$$MSE = E[(\delta - Y)^2] = J(\delta)$$

We want to know $\delta^* = \underset{\delta}{\operatorname{argmin}}\ J(\delta)$

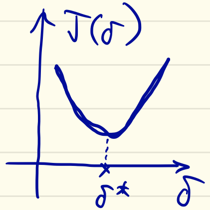
Quest : $\delta^* = ?$

Guess : $\delta^* = E[Y]$

$$\Rightarrow \quad MSE = E[(Y - E[Y])^2] = Var(Y)$$

$$J(\delta) = E[Y^2 - 2\delta Y + \delta^2]$$

$$= \underbrace{E[Y^2]}_{\text{number}} - 2\delta \underbrace{E[Y]}_{\text{number}} + \delta^2$$



To find δ^*

$$\frac{\partial J}{\partial \delta} = 2\delta - 2E[Y] = 0$$

$$\Rightarrow \quad \delta^* = E[Y]$$

(2) Linear estimator $\hat{Y} = aX + b$

$$MSE = E\left[\left(Y - (aX+b)\right)^2\right] = J(a,b)$$

Minimum MSE for linear estimation
$\hat{Y} = a^* X + b^*$ where $(a^*, b^*) = \underset{a,b}{\operatorname{argmin}}\, J(a,b)$

$$J(a,b) = E\left[\left(Y - (aX+b)\right)^2\right]$$

$$= Var(Y - aX - b) + \left(E[Y - aX - b]\right)^2$$

$$= \underbrace{Cov(Y - aX - b,\ Y - aX - b)}_{Cov(Y,Y) - 2a\,Cov(X,Y) + a^2\,Cov(X,X)} + \left(E[Y] - aE[X] - b\right)^2$$

$$\begin{cases} 0 = \dfrac{\partial J(a,b)}{\partial a} \overset{(1)}{=} -2\,Cov(X,Y) + 2a\,Var(X) - 2\left(E[Y] - aE[X] - b\right)E[X] \\ 0 = \dfrac{\partial J}{\partial b} \overset{(2)}{=} -2\left(E[Y] - aE[X] - b\right) \end{cases}$$

From (2) $\Rightarrow$ $b^* = E[Y] - a^* E[X]$; subs into (1) $\Longrightarrow a^* = \ldots$

$$\therefore \quad a^* = \frac{Cov(x,y)}{Var(x)}$$

Hence

MMSE linear estimator

$$\hat{y} = \underbrace{\frac{Cov(x,y)}{Var(x)}}_{a^*} X + \underbrace{E[y] - \frac{Cov(x,y)}{Var(x)} E[x]}_{b^*}$$

$$= \frac{Cov(x,y)}{Var(x)} \left[X - E[x] \right] + E[y]$$

MMSE for linear est: $J(a^*, b^*) = \ldots$

(3) Unconstrained:

Observe $X = u \longrightarrow \hat{Y} = g(u)$

$Y \sim f_{Y|X}(v|u)$

$= E[Y \mid X=u]$

But $E[Y \mid X=u]$ requires $f_{Y|X}(\cdot)$