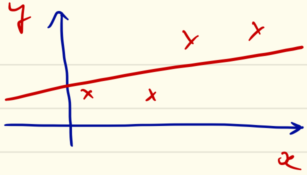
ECE 313: Lecture 34
ML estimation of linear regression
Joint pdfs of functions of random variables (Ch 4.7)
Recognizing independence



Setup: Given training data: $(x_n, y_n)_{n=1}^{N}$

Model:

$y_n -$

"Noise" $y_n = f_{\underbrace{(a,b)}_{\theta}}(x_n) = a x_n + b + W_n$

W_n iid distn $N(0, \sigma^2)$

↑ independent

$\Rightarrow W_n = \underbrace{y_n - (a x_n + b)}_{w_n}$

$$\Rightarrow f_{W_n}(w_n) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{w_n^2}{2\sigma^2}\right)$$

ML

$$f_{(W_1, W_2, \ldots, W_N)}(w_1, w_2, \ldots, w_N) = \prod_{n=1}^{N} f_{W_n}(w_n) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^N \times \exp\left(-\frac{1}{2\sigma^2} \cdot \sum_{n=1}^{N} w_n^2\right)$$

ML estimator

$$\max \; f_{W_1, W_2, \dots, W_N}(w_1, w_2, \dots, w_N)$$

$$\Longleftrightarrow \quad \min \sum_{n=1}^{N} w_n^2 \quad \Longleftrightarrow \quad \min \underbrace{\sum_{n=1}^{N} \left(y_n - (a x_n + b)\right)^2}_{J(a,b)}$$

Hence, under Gaussian "noise" model

ML estimator $\equiv$ min sum of square errors

—

$$\frac{\partial J}{\partial a} = \sum_{n=1}^{N} 2\left(y_n - (a x_n + b)\right) x_n = 0$$

$$\frac{\partial J}{\partial b} = 0$$

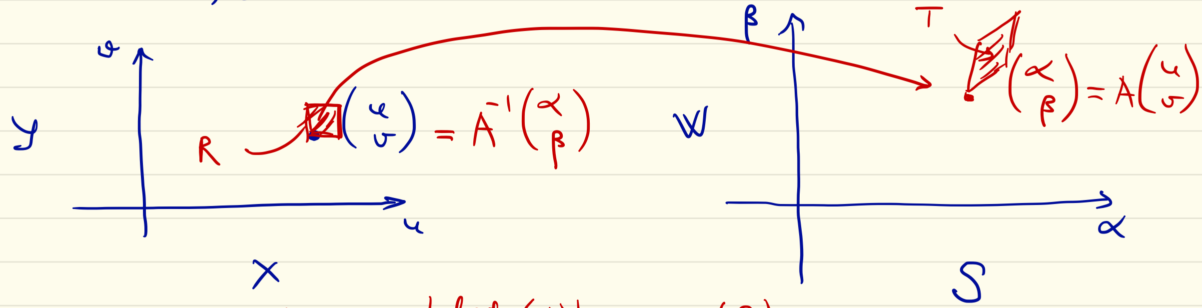
$$\Rightarrow \quad a_{ML} = \frac{\sum_n y_n x_n + \dots}{\sum_n x_n^2}$$

$\circledast$ Prob setting

$$\begin{bmatrix}S\\W\end{bmatrix} = \underbrace{\begin{bmatrix}a & b\\c & d\end{bmatrix}}_{A}\begin{bmatrix}X\\Y\end{bmatrix} \Rightarrow \begin{bmatrix}X\\Y\end{bmatrix} = A^{-1}\begin{bmatrix}S\\W\end{bmatrix}$$

$$\underbrace{X, Y} \xrightarrow{\text{linear tranformation}} S, W$$

$$f_{X,Y}(u,v) \qquad\qquad f_{S,W}(\alpha,\beta) \overset{?}{=}$$



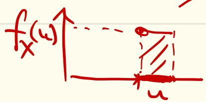
area (T) = |det (A)| area (R)

$$f_{S,W}(\alpha,\beta) = \frac{1}{|\det(A)|}\, f_{X,Y}\left(A^{-1}\binom{\alpha}{\beta}\right)$$

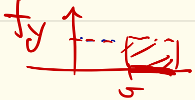
Recall :

$Y = aX \quad (a > 0)$

$v = au \Rightarrow u = \frac{v}{a}$



$$f_X \longrightarrow f_Y(v) = \frac{1}{a}\cdot f_X\left(\frac{v}{a}\right)$$



[**8+4 points**] Suppose X and Y are ~~independent~~ random variables with joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} 2e^{-u}e^{-2v} & \text{if } u \geq 0, v \geq 0 \\ 0 & \text{else.} \end{cases}$$

$$\overbrace{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{A} = ad - bc$$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(a) Find the joint pdf of $S = X + Y$ and $W = Y - X$.

$$\begin{bmatrix} S \\ W \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\begin{bmatrix} S \\ W \end{bmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\det(A) = 2$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$f_{S,W}(\alpha, \beta) = \frac{1}{\det(A)} f_{X,Y}\left(A^{-1}\begin{pmatrix} u \\ v \end{pmatrix}\right)$$

$$= \frac{1}{2} f_{X,Y}\left(\underbrace{\tfrac{1}{2}(\alpha - \beta)}_{u}, \underbrace{\tfrac{1}{2}(\alpha + \beta)}_{v}\right)$$

$$= \begin{cases} e^{-\frac{1}{2}(\alpha - \beta)} e^{-(\alpha + \beta)} & \text{for } \frac{1}{2}(\alpha - \beta) \geq 0; \ \frac{1}{2}(\alpha + \beta) \geq 0 \\ 0 & \text{else} \end{cases}$$

(b) Are S and W independent? Explain.

No, (not a product set) $\longrightarrow$

4.7. [Recognizing **independence**] $\Longleftrightarrow$ ① $f_{X,Y}(u,v) = g(u)\,h(v)$

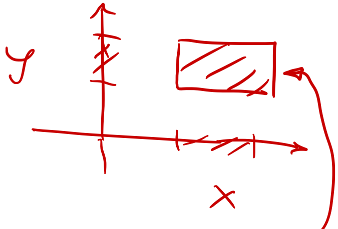
Decide whether X and Y are independent for each of the following three joint pdfs. If they are independent, identify the marginal pdfs f_X and f_Y. If they are not, give a reason why.

② Support = product region

$= e^{-u^2} \cdot e^{-v^2}$

Y (a) $f_{X,Y}(u,v) = \begin{cases} \frac{4}{\pi} e^{-(u^2+v^2)} & u, v \geq 0 \\ 0 & \text{else.} \end{cases}$

Y (b) $f_{X,Y}(u,v) = \begin{cases} -\frac{ln(u)v^2}{21} & 0 \leq u \leq 1, 1 \leq v \leq 4 \\ 0 & \text{else.} \end{cases}$



N (c) $f_{X,Y}(u,v) = \begin{cases} \frac{(96)u^2v^2}{\pi} & u^2 + v^2 \leq 1 \\ 0 & \text{else.} \end{cases}$ not product set