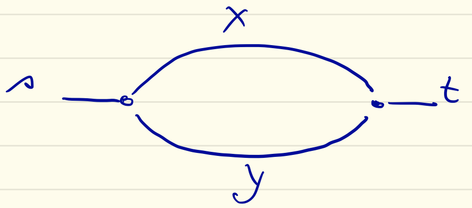
ECE 313: Lecture 32
More problems involving joint densities (Ch 4.6)



Ex 1: X & Y are lifetime of each brand, independent

Z is lifetime of the whole system

$$Z = \max\{X, Y\}$$

Prob: Find pdf of Z given pdf / CDF of X & Y

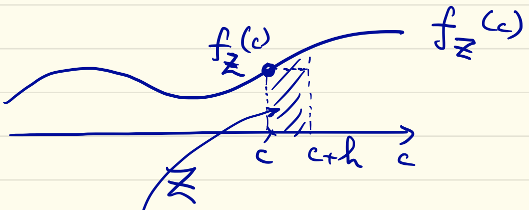
Solution: 1. Find CDF of Z

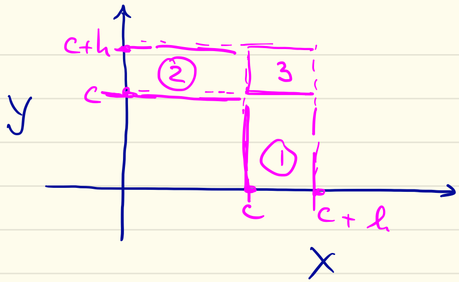
$$F_Z(c) = P(Z \leq c) = P(\max(X,Y) \leq c)$$
$$= P(X \leq c, Y \leq c)$$
$$\overset{\text{(independent)}}{=} P(X \leq c)\, P(Y \leq c)$$
$$= F_X(c)\, F_Y(c)$$

2. Take derivative: to get pdf

$$f_Z(c) = \frac{d}{dc}\left(F_X(c)\, F_Y(c)\right) = F_X'(c)\, F_Y(c) + F_X(c)\, F_Y'(c)$$
$$= f_X(c)\, F_Y(c) + F_X(c)\, f_Y(c)$$

Intuition : $Z = \max\{X, Y\}$





In general:

$$\text{Area} = P[c \le Z < c+h] = f_Z(c)\,h + o(h)$$

$$\approx f_Z(c) \cdot h$$

where $\lim_{h \to 0} \frac{o(h)}{h} = 0$

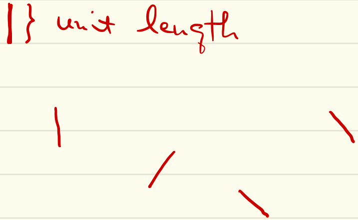
For $Z = \max\{X, Y\}$

$$f_Z(c) \cdot h = P(\underbrace{c \le Z < c+h}_{\text{Then either } X \text{ or } Y \text{ has to be in } [c,\, c+h)}) = P(\underbrace{c \le X < c+h,\ Y \le c}_{f_X(c)\cdot h \,\cdot\, F_Y(c)})$$

$$+\ P(\underbrace{X \le c,\ c \le Y < c+h}_{F_X(c)\ f_Y(c)\ h})$$

$$+\ P(\underbrace{c \le X < c+h,\ c \le Y < c+h}_{f_X(c)\ f_Y(c)\ h^2})$$

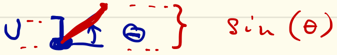
Ex 2: Buffon's needle prob.



Throw a unit length needle randomly on a paper with unit grid lines

$$P\{\underbrace{\text{needle cross a line}}_{A}\}$$

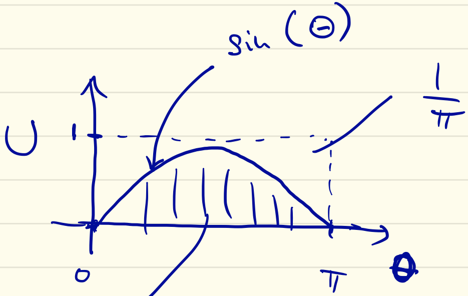
needle $\begin{cases} U \sim \text{Unif}[0,1] \\ \Theta \sim \text{Unif}[0,\pi] \end{cases}$ independent



$$P(A) = P(\sin\Theta > U)$$

$$= \text{Area} * \frac{1}{\pi}$$

$$= \frac{2}{\pi}$$



Ex 3 : (Linear regression) ← or basic Machine Learning

new

$$\underbrace{(x_n, y_n)_{n=1}^{N}}_{\text{Training data}} \Rightarrow y_n = a\, x_n + b \Longrightarrow \underbrace{y = a x + b}_{\text{testing data}}$$

<u>Estimation</u> → (a, b) → Inference → testing data

<u>Model</u> : For $n = 1, 2, \ldots, N$

$$y_n = a\, x_n + b + W_n$$

"noise"

$$W_n \overset{i.i.d}{\sim} \mathcal{N}(0, \sigma^2)$$

independent identically distrib.