# ECE 313: Lecture 24
## ML parameter estimation for continuous type random variables (Ch. 3.7)
## The distribution of a function of a random variable (Ch 3.8.1)

unknown distribution

$$\boxed{\text{pdf: } f_\theta} \longrightarrow \quad \text{Observation } \{X_i = x_i\} \quad i = 1, 2, \ldots$$

given

$\theta$

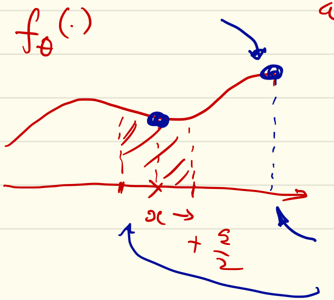
$\hat{\theta} = ?$ $\longleftarrow$ estimate

$X$ is assumed to have pdf $f_\theta$

Observing $X = x \longrightarrow f_\theta(x) = P\{X = x\}$ discrete-type

In continuous-type: $f_\theta(x) \neq P\{X = x\}$

assume $f_\theta(x)$ is continuous func.




more likely than

$$f_\theta(x) = F'(x) \quad \text{where } F(x) \overset{\text{CDF}}{=} P\{X \le x\}$$

$$= \lim_{\varepsilon \to 0} \frac{F(x + \frac{\varepsilon}{2}) - F(x - \frac{\varepsilon}{2})}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( P\{x - \tfrac{\varepsilon}{2} < X \le x + \tfrac{\varepsilon}{2}\} \right)$$

unknown

**3.23. [ML parameter estimation of the rate of a Poisson process]**
Calls arrive in a call center according to a Poisson process with arrival rate $\lambda$ (calls/minute). Derive the maximum likelihood estimate $\widehat{\lambda}_{ML}$ if $\underbrace{k \text{ calls are received in a } T \text{ minute interval}}_{\text{observation}}$.

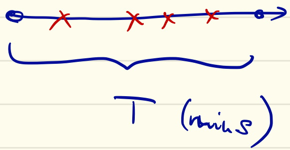
estimate

Let $X$ be # calls in $T$ min interval

$$X \sim \mathrm{Poi}(\lambda T)$$

(Recall: $E[X] = \lambda T$

rate $\dfrac{E[X]}{T} = \lambda$ (calls/min) )




$$P\{X = k\} = \underbrace{e^{-(\lambda T)} \cdot \frac{(\lambda T)^k}{k!}}_{J(\lambda)} = \frac{1}{k!} T^k \underbrace{e^{-(\lambda T)} \cdot \lambda^k}$$

In our estimation prob., $T$ & $k$ are given, view the above likelihood as a func of $\lambda$

$$\hat{\lambda}_{ML} = \underset{\lambda}{\operatorname{argmax}}\ J(\lambda)$$

To do this $0 = \dfrac{dJ(\lambda)}{d\lambda} = e^{-\lambda T} k \lambda^{k-1} + e^{-\lambda T} \cdot (-T) \cdot \lambda^k$

$$\Rightarrow \quad k + (-T)\hat{\lambda}_{ML} = 0 \quad \Longleftrightarrow \quad \hat{\lambda}_{ML} = \frac{k}{T}$$

given $\quad X \xrightarrow{\;Y = g(X)\;} Y \quad$ derive ?

| | $X$ | | $Y$ |
|---|---|---|---|
| pdf: | $f_X$ | pmf/pdf: | $f_Y$ |
| CDF | $F_X$ | CDF: | $F_Y$ |

① $f_X \to F_X$ ② $F_X \to F_Y$ ③ $F_Y \to f_Y$

(*) When $Y$ is continuous-type $\qquad f_Y(u) = F'_Y(u)$

Ex: $\quad Y = aX + b \qquad a, b$ constants, $a > 0$

$$X \sim f_X(u) \longrightarrow Y \sim f_Y(v) = \frac{1}{a} f_X\Big(\underbrace{\frac{v-b}{a}}_{u}\Big)$$

(*) When $Y$ is discrete-type

pmf $\quad f_Y(k) = P\{Y = k\} = P\{X \in S_k\} = \int_{u \in S_k} f_X(u)\,du$

where $\quad S_k = \{x : g(x) = k\}$