ECE 313: Lecture 23
The central limit theorem and Gaussian approximation (Ch. 3.6.3)

Recall: $X \sim \mathcal{N}(\mu, \sigma^2)$

$$f_X(u) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{(u-\mu)^2}{2\sigma^2}\right)$$

F_X

Standard

$$\tilde{X} = \frac{X-\mu}{\sigma} \sim \mathcal{N}(0; 1)$$

$$P\left(\frac{X-\mu}{\sigma} \le u\right) = P(\tilde{X} \le u) = F_{\tilde{X}}(u) = \Phi(u)$$

look-up table

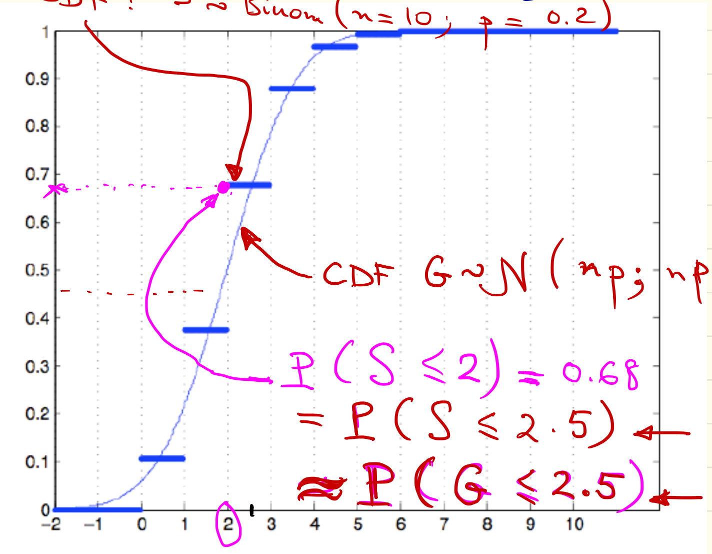
$$P\left(\frac{X-\mu}{\sigma} \ge u\right) = P(\tilde{X} \ge u) = Q(u) = 1 - \Phi(u)$$

$P(X \le v)$ $\quad x = v \quad$ $\tilde{X} = \frac{v-\mu}{\sigma} = u$

Central limit theorem (CLT): X_i are independent identically distributed with $E[X_i] = \mu$, $Var[X_i] = \sigma^2$

$$\underbrace{X_1 + X_2 + \dots + X_n}_{S} \approx N(\underbrace{n\mu}_{E[S]}, \underbrace{n\sigma^2}_{Var[S]})$$

CDF: $S \sim Binom(n=10; p=0.2)$

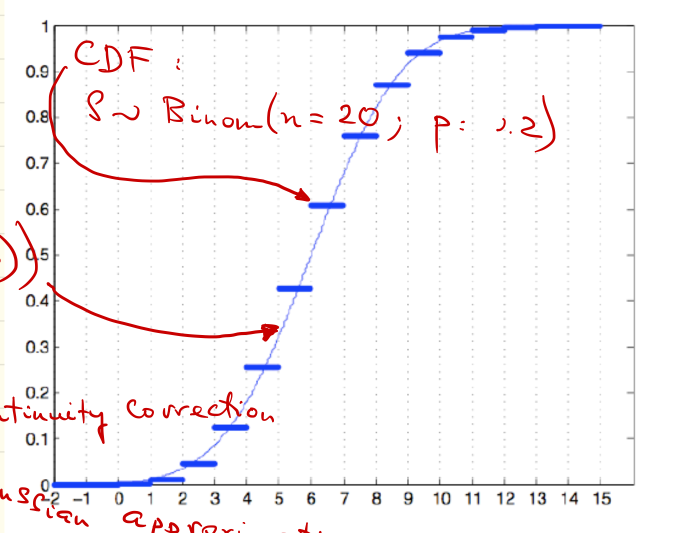


CDF $G \sim N(np; np(1-p))$

$P(S \leq 2) = 0.68$

$= P(S \leq 2.5)$ ← continuity correction

$\approx P(G \leq 2.5)$ ← Gaussian approximation



CDF: $S \sim Binom(n=20; p=0.2)$

Ex: $S = X_1 + \dots + X_n$ $\quad X_i \sim Bern(p)$

$S \sim Binom(n, p)$

Gauss approx: $S \approx \overbrace{N(np, np(1-p))}^{G}$

```python
def binomial(n, p):
    pmf = np.zeros(n+1)
    for i in range(n+1):
        pmf[i] = scipy.special.binom(n, i) * p**i * (1-p)**(n-i)

    return pmf
```

```python
n = 100
p = 3/10
pmf_b = binomial(n, p)
plt.stem(pmf_b)
plt.title('pmf of Binomial($n, p$) for $n = {}, p = {}$'.format(n, p));
```

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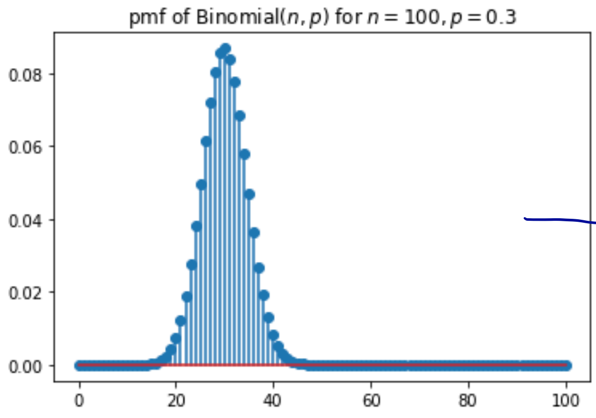
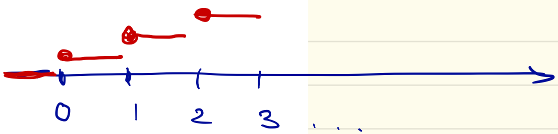
CDF





$S \sim \text{Binom}(n=100;\ p=0.5)$

$G \approx N(\mu = np;\ \sigma^2 = np(1-p))$

$\mu = 50 \qquad \sigma = 5$

3.19. **[Heads minus tails]**
Suppose a fair coin is flipped 100 times, and A is the event:
$A = \{\ |\underbrace{(\text{number of heads})}_{S} - \underbrace{(\text{number of tails})}_{100-S}| \geq 10\ \}$.

(a) Let S denote the number of heads. Express A in terms of S. Specifically, identify which values of S make A true.

(b) Using the Gaussian approximation with the continuity correction, express the approximate value of $P(A)$ in terms of the Q function.

a)
$$A = \{\, |S - (100 - S)| \geq 10 \,\}$$
$$= \{\, |2S - 100| \geq 10 \,\}$$
$$= \{\, 2S - 100 \geq 10 \,\} \cup \{\, 2S - 100 \leq -10 \,\}$$
$$= \{\, S \geq 55 \,\} \cup \{\, S \leq 45 \,\}$$

b)
$$P(A) = P(S \geq 55) + P(S \leq 45)$$

Continuity correction:
$$P(S \geq \underbrace{55 - 0.5}) + P(S \leq \underbrace{45 + 0.5})$$
$$= P\left(\frac{S-50}{5} \geq \frac{54.5-50}{5}\right) + P\left(\frac{S-50}{5} \leq \frac{45.5-50}{5}\right)$$

Gaussian approximation:
$$\approx Q(0.9) + \Phi(-0.9)$$
$$= 2\,Q(0.9)$$