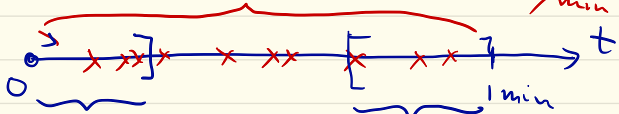
ECE 313: Lecture 22
Scaling rule for pdfs (Ch. 3.6.1)
Gaussian (normal) distribution (e.g. using Q and Phi functions) (Ch. 3.6.2)

But first… let's do one more Poisson process problem:

3.11. **[Poisson process]**
A certain application in a cloud computing system is accessed on average by 15 customers per minute. Find the probability that in a one minute period, three customers access the application in the first ten seconds and two customers access the application in the last fifteen seconds. (Any number could access the system in between these two time intervals.)

rate $\lambda = 15/\text{min}$

X_1 = # customers → first 10 sec $= \frac{1}{6}$ min



X_2 = # customers → last 15 sec $= \frac{1}{4}$ min

$X_1 \sim \text{Poi}(\underbrace{\lambda \cdot \tfrac{1}{6}}_{\frac{15}{6}})$

$X_2 \sim \text{Poi}(\lambda \cdot \tfrac{1}{4} = \tfrac{15}{4})$

X_1 & X_2 are independent

We want:

$P\{X_1 = 3, X_2 = 2\}$

$= P\{X_1 = 3\} \cdot P\{X_2 = 2\}$

$= e^{-\frac{15}{6}} \frac{(15/6)^3}{3!} \cdot e^{-\frac{15}{4}} \frac{(15/4)^2}{2!}$

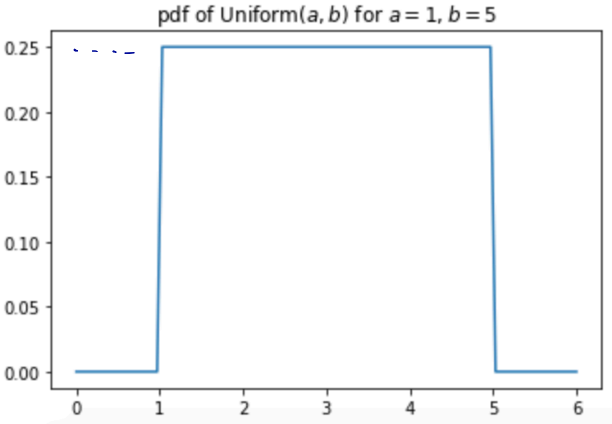
Uniform(a, b)

```python
def uniform(x, a, b):
    return 1/(b-a)*(x >= a)*(x <= b)
```

$$pdf_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{else} \end{cases}$$

```python
a = 1
b = 5
x = np.linspace(0, 6, 100)
fu = uniform(x, a, b)
plt.plot(x, fu)
plt.title('pdf of Uniform($a, b$) for $a = {}, b = {}$'.format(a, b));
```

$\frac{1}{b-a}$



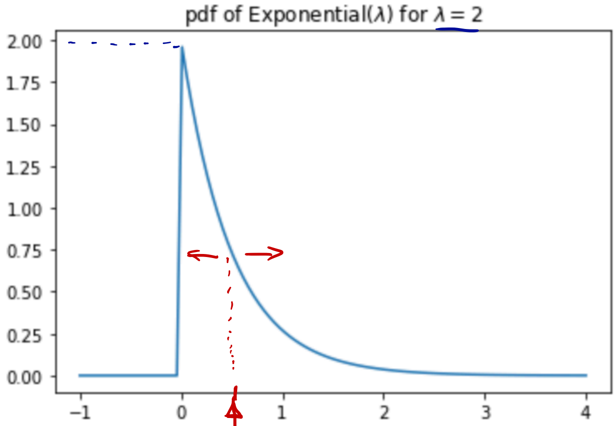
a b

Exponential(λ)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{else} \end{cases}$$

```python
def exponential(x, lamb):
    return lamb * np.exp(-lamb*x) * (x >= 0)
```

```python
lamb = 2
x = np.linspace(-1, 4, 100)
fe = exponential(x, lamb)
plt.plot(x, fe)
plt.title('pdf of Exponential($\lambda$) for $\lambda = {}$'.format(lamb));
```



$$E[X] = \frac{1}{\lambda} \qquad \sqrt{Var[X]} = \frac{1}{\lambda}$$

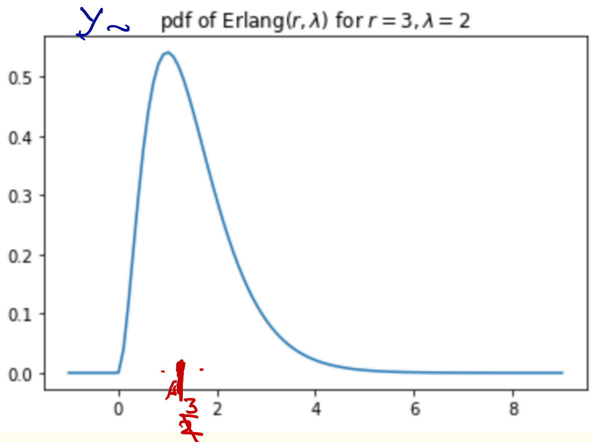
$$Y = X_1 + X_2 + \dots + X_r$$

$Y \sim$ **Erlang(r, λ)**

$$X_i \sim \text{Exponential}(\lambda)$$

```python
def erlang(x, r, lamb):
    return lamb**r * x**(r-1) * np.exp(-lamb*x) / math.factorial(r-1) * (x >= 0)
```

```python
r = 3
lamb = 2
x = np.linspace(-1, 9, 100)
fer = erlang(x, r, lamb)
plt.plot(x, fer)
plt.title('pdf of Erlang($r, \lambda$) for $r = {}, \lambda = {}$'.format(r, lamb));
```



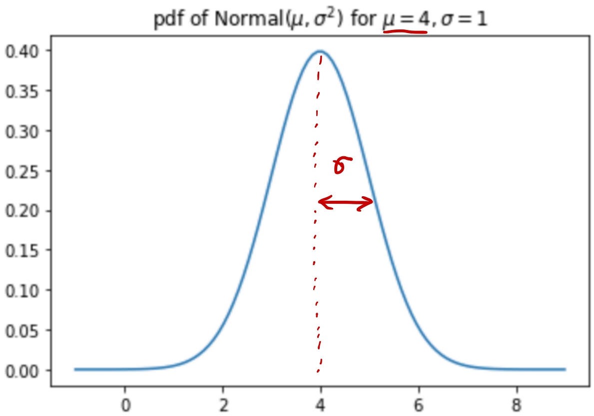
$$E[Y] = E[X_1 + \dots + X_r] = E[X_1] + \dots + E[X_r] = \frac{r}{\lambda}$$

$X \sim$ **Normal**(μ, σ^2)

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

```python
def normal(x, mu, sigma):
    return 1/np.sqrt(2*np.pi*sigma**2) * np.exp(-(x - mu)**2 / (2*sigma**2))
```

```python
mu = 4
sigma = 1
x = np.linspace(-1, 9, 100)
fn = normal(x, mu, sigma)
plt.plot(x, fn)
plt.title('pdf of Normal($\mu, \sigma^2$) for $\mu = {}, \sigma = {}$'.format(mu, sigma));
```



$E[X] = \mu$

$Var[X] = \sigma^2$

Suppose that X is a r.v. that has pdf $f_X(x)$

Consider $Y = aX + b$ ($a > 0$, b are constants)

What is the pdf of Y ?

① $f_Y(y) \overset{\times}{=} f_X\left(\frac{y-b}{a}\right)$ by $y = ax+b$, $x = \frac{y-b}{a}$

② Correct way:

$$\begin{array}{ccc} \overset{\text{pdf}}{f_Y} & \longleftarrow & \overset{\text{CDF}}{F_Y} \\ & & \uparrow \\ f_X & \longrightarrow & F_X \end{array}$$

$$\begin{aligned} F_Y(c) &= P\{Y \le c\} \\ &= P\{aX + b \le c\} \\ &= P\left\{X \le \frac{c-b}{a}\right\} \quad (\text{since } a > 0) \\ &= F_X\left(\frac{c-b}{a}\right) \end{aligned}$$

$$f_Y(c) = F_Y'(c) = \frac{d}{dc} F_X\left(\frac{c-b}{a}\right) = \frac{1}{a} F_X'\left(\frac{c-b}{a}\right) = \frac{1}{a} f_X\left(\frac{c-b}{a}\right)$$

* Suppose X is r.v. $\quad E[X] = \mu$

$\quad Var[X] = \sigma^2$

Let $\tilde{X} = \dfrac{X-\mu}{\sigma}$ or $X = \sigma\tilde{X} + \mu$

$$E[\tilde{X}] = \frac{E[X]-\mu}{\sigma} = 0$$

$$Var[\tilde{X}] = \frac{1}{\sigma^2} Var[X] = 1$$

Thus $\underline{\tilde{X} \text{ is standardized version of } X}$

$$X \sim N(\mu, \sigma^2) \longrightarrow \tilde{X} = \frac{X-\mu}{\sigma} \sim N(0,1)$$

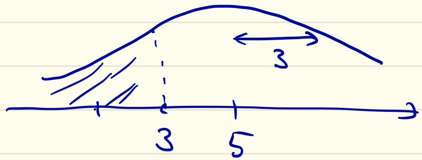
Normal
(Gaussian)

$$f_{\tilde{X}}(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

$$\Phi(u) \overset{\text{CDF}}{=} F_{\tilde{X}}(u) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$

$$Q(u) = 1 - \Phi(u) = \Phi(-u)$$

Ex: $X \sim N(\overset{\mu}{5}, \overset{\sigma^2}{9})$



$$P\{X \leq 3\}$$

$$= P\left\{ \underbrace{\frac{X - \overset{\mu}{5}}{\underset{\sigma}{3}}}_{\tilde{X}} \leq \underbrace{\frac{3-5}{3}}_{-\frac{2}{3}} \right\}$$

$$= \Phi\left(-\frac{2}{3}\right)$$

$$= Q\left(\frac{2}{3}\right)$$