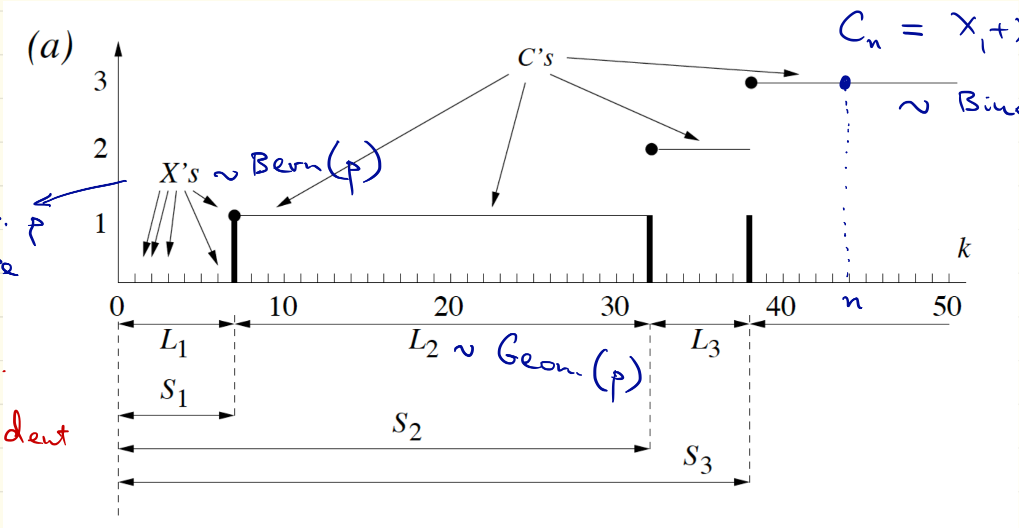
ECE 313: Lecture 21
Poisson processes (Ch 3.5)
Erlang distribution (Ch 3.5.3)

Bernoulli process



$C_n = X_1 + X_2 + \dots + X_n \sim \text{Binom}(n, p)$

X's $\sim \text{Bern}(p)$

For each i:

$$X_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{else} \end{cases}$$

$X_1, X_2, X_3, \dots$ are independent

$L_2 \sim \text{Geom.}(p)$

n

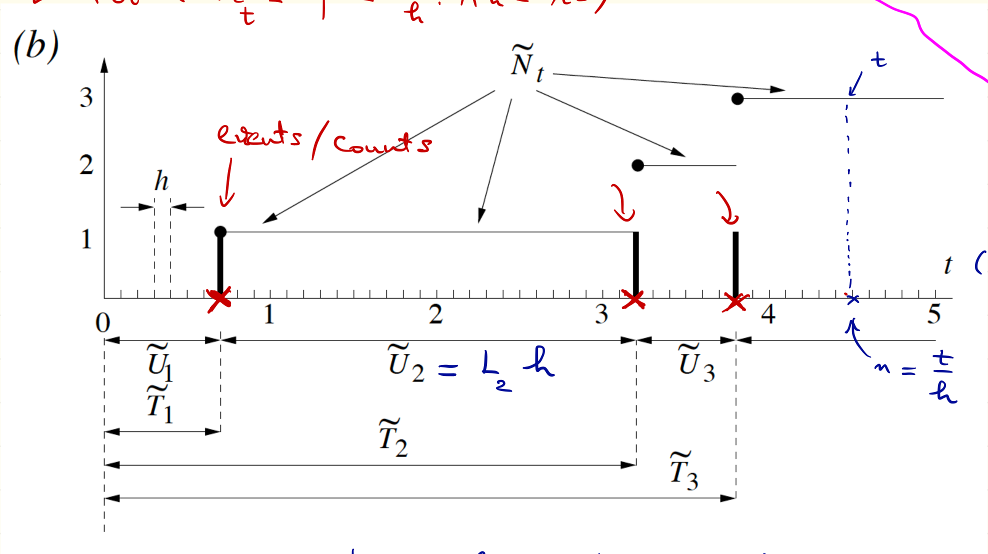
Time-scaled Bernoulli process $\xrightarrow{h \to 0}$ Poisson process

$\tilde{N}_t \sim \text{Binom}\left(n = \frac{t}{h},\ p = \lambda h\right)$

$\xrightarrow[h \to 0]{}$ Poi $\left(\lambda_t = np = \frac{t}{h} \cdot \lambda h = \lambda t\right)$

Fact: $\lim_{h \to 0} \underbrace{(1-\lambda h)^{\frac{1}{h}}}_{f(h)} = e^{-\lambda}$

Since $\ln[f(h)] = \frac{\ln(1-\lambda h)}{h} \xrightarrow{h \to 0} -\lambda$



Let T be the amount of time from $t=0$, till seeing the next event

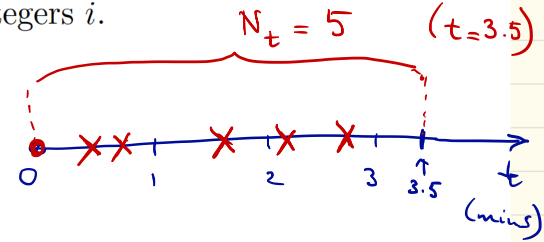
We have: $T = Lh$, $L \sim \text{Geom}(p)$ $\left\{\begin{array}{l}\text{Rate } \lambda \\ p = \lambda h\end{array}\right.$

$P\{T > c\} = P\{L > \frac{c}{h}\} \approx (1-p)^{\frac{c}{h}}$

$= \left(1 - \frac{\lambda}{h}\right)^{\frac{c}{h}} \xrightarrow[h \to 0]{} e^{-\lambda c}$ Or $T \sim \text{Exp}(\lambda)$

$\sim \mathrm{Poi}(\lambda t)$, where $\lambda = 2$

Example 3.5.4 Calls arrive to a cell in a certain wireless communication system according to a Poisson process with arrival rate $\lambda = 2$ calls per minute. Measure time in minutes and consider an interval of time beginning at time $t = 0$. Let N_t denote the number of calls that arrive up until time t. For a fixed $t > 0$, the random variable N_t is a Poisson random variable with parameter $2t$, so it's pmf is given by $\boxed{P\{N_t = i\} = \frac{e^{-2t}(2t)^i}{i!}}$ for nonnegative integers i.



(a) Find the probability of each of the following six events:

E_1=“No calls arrive in the first 3.5 minutes.”

$P(E_1) = P(N_{3.5} = 0)$ $\qquad e^{-7}\cdot\frac{(7)^0}{0!}$

E_2=“The first call arrives after time $t = 3.5$.”

$P(E_2) = P(E_1)$

E_3=“Two or fewer calls arrive in the first 3.5 minutes.”

$P(E_3) = P\{N_{3.5} = 0\} + \underbrace{P\{N_{3.5} = 1\}}_{= e^{-7}\cdot\frac{7^1}{1!}} + \underbrace{P\{N_{3.5} = 2\}}_{= e^{-7}\cdot\frac{7^2}{2!}}$

E_4=“The third call arrives after time $t = 3.5$.”

E_5=“The third call arrives after time t.” (for general $t > 0$)

E_6=“The third call arrives before time t.” (for general $t > 0$)

(b) Derive the pdf of the arrival time of the third call.

(c) Find the expected arrival time of the tenth call?