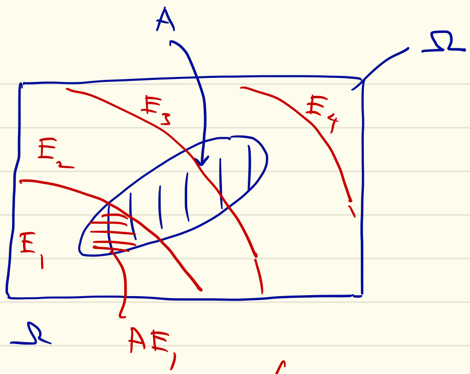
ECE 313: Lecture 14
Law of total probability
Bayes formula



$P(A)$

Let $E_1, E_2, \dots,$ be a partition of Ω

$$\bigcup E_i = \Omega \qquad E_i E_j = \phi$$

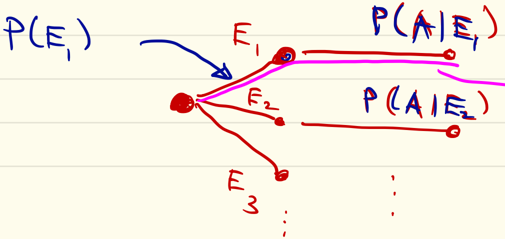
(Notation $AB = A \cap B$)

$$\Rightarrow \quad A \underset{\text{partition}}{=} AE_1 + AE_2 + \dots$$

Law of total probability $\Rightarrow$

$$P(A) = P(AE_1) + P(AE_2) + \dots$$

$$= P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + \dots$$



$$\sum \underbrace{P(E_i)\,P(A|E_i)}_{i} = P(A)$$

Ex : 3 coins : $\begin{cases} HH \leftarrow E_1 \\ TT \leftarrow E_2 \\ HT \leftarrow E_3 \end{cases}$

E_i

{ Pick one coin, throw
see a H

R_1

Throw again (same coin)
$P(\text{get } H) = ?$

R_2

$$P(R_1) = P(E_1)\,P(R_1 \mid E_1) + P(E_2)\,P(R_1 \mid E_2) + P(E_3)\,P(R_1 \mid E_3)$$

$$= \frac{1}{3}\cdot 1 + \frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2}$$

$$P(R_2 \mid R_1) = \frac{P(R_1 R_2)}{P(R_1)} = \frac{5/12}{1/2} = \frac{5}{6}$$

And $P(R_1 R_2) = \sum_{i=1}^{3} P(E_i)\,P(R_1 R_2 \mid E_i) = \sum_{i=1}^{3} P(E_i)\,P(R_1 \mid E_i)\cdot P(R_2 \mid E_i)$

$$= \frac{1}{3}\cdot 1^2 + \frac{1}{3}\cdot 0^2 + \frac{1}{3}\cdot\left(\frac{1}{2}\right)^2 = \frac{1}{3}\cdot\frac{5}{4}$$

$$P(A) \overset{\text{Total prob}}{=} \sum_i P(E_i) \underbrace{P(A \mid E_i)}$$

Bayes formula:

$$P(E_i \mid A) = \frac{P(E_i A)}{P(A)} = \frac{P(E_i)\, P(A \mid E_i)}{\sum_i P(E_i)\, P(A \mid E_i)}$$

Ex: (Cont.) Suppose we get 2 H (R_1, R_2)

$$P(E_1 \mid R_1 R_2) = \frac{\frac{1}{3} \cdot (1)^2}{\frac{1}{3} \cdot (1)^2 + \frac{1}{3} \cdot (0)^2 + \frac{1}{3} \cdot \left(\frac{1}{2}\right)^2} = \frac{\frac{1}{3}}{\frac{1}{3} \cdot \frac{5}{4}}$$

$$= \frac{4}{5}$$

Extend to calculate mean, ...

$$E[X] \overset{def}{=} \sum_k k \, P(\{X = k\})$$

$$E[X \mid E_i] \overset{def}{=} \sum_k k \, P(\{X = k\} \mid E_i)$$

Then
$$\sum_i P(E_i)\, E[X \mid E_i] = \sum_i P(E_i) \overbrace{\sum_k k \, P(\{X = k\} \mid E_i)}$$

$$= \underbrace{\sum_k k \underbrace{\sum_i P(E_i)\, P(\{X = k\} \mid E_i)}_{P(\{X = k\})}}_{E[X]}$$

Recall : $L \sim \text{Geometric}(p)$

$$E[L] = \underbrace{P(\underbrace{\text{first throw} = H}_{E_1})\, E[L \mid E_1]}_{} + P(\{\underbrace{\text{first throw} = T}_{E_2}\})\, \underbrace{E[L \mid E_2]}_{}$$

$$E[L] = p \cdot 1 + (1-p)\,(E[L] + 1)$$

$$\Rightarrow \quad E[L] = \frac{1}{p}$$