ECE 313: Lecture 12
Maximum likehood parameter estimation
Markov and Chebychev inequalities

Ex1: Suppose we a coin $\begin{cases} 1 \ (H) & \text{w.p. } p \\ 0 \ (T) & \text{w.p. } 1-p \end{cases} \sim \text{Bern}(p)$

Throw 10 trials, record outcome

$$\underbrace{\{X_i\}_{i=1}^{10} = (0, 0, 1, 0, 0, 0, 1, 1, 0, 0)}$$

Q: 1 Find an estimate of p as $\hat{p}$

2 Find / Determine the confidence of this $\hat{p}$

Ex 1(b) "Polling"

Ex 1(c) System realiability testing

Go back to Ex 1:

$$P\{\ \cdot\ \} = P\{X_1 = 0\} \cdot P\{X_2 = 0\} \cdot P\{X_3 = 1\} \cdots$$

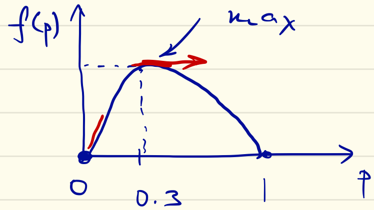
$$= (1-p) \cdot (1-p) \cdot p \cdots (1-p) \cdots$$

$$= p^{\#\text{ of } 1\text{'s}} (1-p)^{\#\text{ of } 0\text{'s}} = p^3 (1-p)^7$$

Let $f(p) = P_p\{(X_1, X_2, \dots, X_n) = (0, 0, 1, \dots)\}$
$= \boxed{p^3 (1-p)^7}$
is the likelihood func.

Maximum likelihood estimator

$$\hat{p}_{ML} = \arg\max_{0 \le p \le 1} f(p)$$



$$f(0) = 0$$
$$f(1) = 0$$
$$f(0.3) = (0.3)^3 (0.7)^7$$

Consider a more general form $\left\{ \begin{array}{l} n \text{ trials} \\ \Rightarrow k \text{ of '1'} \end{array} \right.$

$$f(p) = p^k (1-p)^{n-k}$$

$$\frac{df}{dp} = k p^{k-1} (1-p)^{n-k} + p^k (n-k)(1-p)^{n-k-1} \cdot (-1)$$

$$= p^{k-1} (1-p)^{n-k-1} \left[\underbrace{k(1-p) - p(n-k)}_{=0} \right]$$

$$\frac{df}{dp} = 0 \iff$$

$$\frac{df}{dp} = 0 \iff k(1-p) - p(n-k) = 0$$

$$\iff k - kp - (n-k)p = 0$$

$$\iff np = k$$

$$\iff \cancel{k} \quad p = \frac{k}{n}$$

Hence $\hat{p}_{ML} = \frac{k}{n}$

prob of "H" is p (unknown)

Ex 2: Throw a coin n times

Observe $C = k$ heads

$\hat{p} = ?$ $\quad C \sim \text{Binom}(n, p)$ unknown

$$g(p) = \mathbb{P}\{C = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\arg\max_p g(p) = \arg\max_p p^k (1-p)^{n-k} = \frac{k}{n}$$

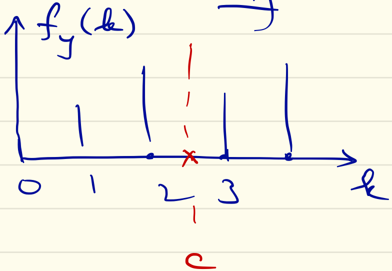
$\star$ Markov inequality:

Let Y be a <u>non negative</u> discrete rand variable (rv)

For a fixed $c > 0$:

$$P\{Y \geq c\} \leq \frac{E[Y]}{c}$$

Proof:



$$E[Y] = \sum_{k \geq 0} k \cdot P\{Y = k\}$$

$$= \underbrace{\sum_{0 \leq k < c} k\, P\{Y = k\}}_{\geq 0} + \underbrace{\sum_{k \geq c} k\, P\{Y = k\}}_{\geq c \cdot \underbrace{\sum_{k \geq c} P\{Y = k\}}_{P\{Y \geq c\}}}$$

* Chebychev inequality:

Let $Y = |X - E[X]|^2$ nonneg r.v.

$$P\{Y \geq d^2\}$$

$$= P\{|X - E[X]| \geq d\}$$

from Markov

$$\leq \frac{E[|X - E[X]|^2]}{d^2} = \frac{Var(X)}{d^2}$$

Ex: X r.v. with $E[X] = 3.5$

$Var[X] = 0.01$

$$\Rightarrow P\{|X - 3.5| \geq 0.2\} \leq \frac{0.01}{0.2^2}$$

$$= \frac{1}{4}$$