ECE 313: Lecture 11
Poisson distribution -- a limit of binomial distribution

<u>Binomial</u>: $X \sim \text{Binomial}(n, p)$

$$P(\{X = k\}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Ex 1: Throw a dice 10 times. Let X be the number of "6" 

$X \sim \text{Binomial}(10, \frac{1}{6})$

$$P(\{X = 3\}) = \binom{10}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^7$$

Note $a ** b = a^b$

In PrairieLearn OR Python $=$ $(10 * 9 * 8)/(3*2*1) * ((1/6)**3) * ((5/6)**7)$

Ex: In a tweet of 100 words, the prob of making a misspelled word is $p = 0.001$

X is the number of misspelled words

$X \sim \text{Binomial}(100, 0.001)$ $\quad E[X] = np = 0.1$

$$P \sim \text{Poisson}(\lambda)$$

$$f_P(k) = P(\{X = k\}) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2, \dots$$

Compare to:

$$B \sim \text{Binomial}(n, p)$$

$$f_B(k) = P(\{B = k\}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Let $\lambda = np$

$$f_B(k) = \frac{n(n-1)\dots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k \left(\frac{n-\lambda}{n}\right)^{n-k}$$

$$= \frac{\lambda^k}{k!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\downarrow\, e^{-\lambda}} \underbrace{\left[\frac{n(n-1)\dots(n-k+1)}{n^k}\right]}_{\downarrow\, 1} \underbrace{\left[\left(1 - \frac{\lambda}{n}\right)^{-k}\right]}_{\downarrow\, 1}$$

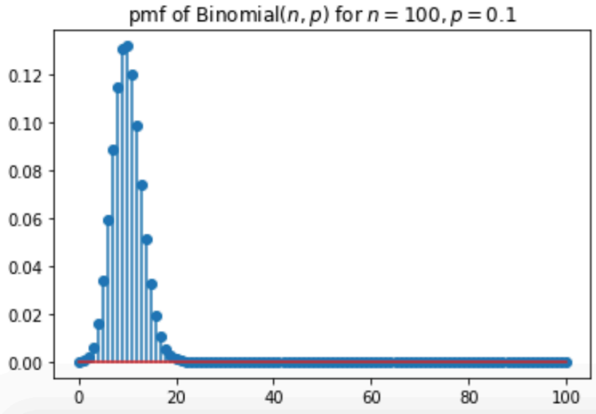
$$\overset{n \to \infty}{=} \approx f_P(\lambda)$$

Binomial(n, p)

```
def binomial(n, p):
    pmf = np.zeros(n+1)
    for i in range(n+1):
        pmf[i] = scipy.special.binom(n, i) * p**i * (1-p)**(n-i)

    return pmf
```

```
n = 100
p = 1/10
pmf_b = binomial(n, p)
plt.stem(pmf_b)
plt.title('pmf of Binomial($n, p$) for $n = {}, p = {}$'.format(n, p));
```

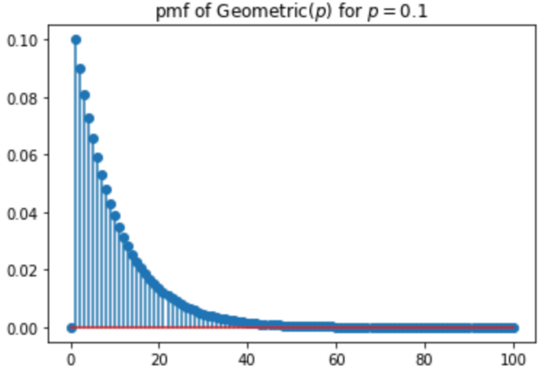


Geometric(p)

```
def geometric(p, n = 100):
    pmf = np.zeros(n+1)
    for i in range(1,n+1):
        pmf[i] =  (1-p)**(i-1) * p

    return pmf
```

```
p = 1/10
pmf_g = geometric(p)
plt.stem(pmf_g)
plt.title('pmf of Geometric($p$) for $p = {}$'.format(p));
```



$X \sim \text{Poisson}(\lambda)$

Poisson(λ)

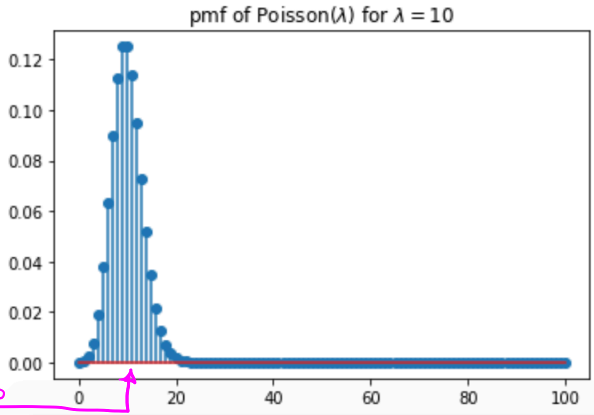
```python
def poisson(lamb, n = 100):
    pmf = np.zeros(n+1)
    for i in range(n+1):
        pmf[i] =  lamb**i * math.exp(-lamb) / math.factorial(i)

    return pmf
```

$= f_X(k)$

$$\sum_{k=0}^{\infty} f_X(k) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

```python
lamb = 10
pmf_p = poisson(lamb)
plt.stem(pmf_p)
plt.title('pmf of Poisson($\lambda$) for $\lambda = {}$'.format(lamb));
```



$E[X] = \lambda = 10$

$$E[X] = \sum_{k=0}^{\infty} k \cdot f_X(k)$$

$$= \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\cancel{k} \lambda^k}{(k-1)!} \rightarrow \frac{\lambda \cdot \lambda^{k-1}}{(k-1)!}$$

$$= \lambda$$

Recall: $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$