ECE 313: Lecture 10
Bernoulli process
Connections between Bernoulli, binomial, and geometric distributions

Game: Throw a fair dice. Get an award if 6 shows up

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>>> D = [random.randint(1,6) for i in range(20)]
>>> D
[6, 3, 3, 2, 2, 3, 3, 6, 6, 3, 1, 5, 3, 6, 4, 5, 2, 2, 2, 4]
```

$$X_i = \begin{cases} 1 & \text{if } D_i = 6 \\ 0 & \text{else} \end{cases}$$

← one outcome after 20 throws

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
X	1	0	0	0	0	0	0	1	1	0	0	0	0	1	0	0
C	1	1	1	1	1	1	1	2	3	3	3	3	3	4	4	4
L	$L_1 = 1-0 = 1$							$L_2 = 8-1 = 7$	$L_3 = 1$					$L_4 = 14-9 = 5$		

$$X \Longrightarrow \text{pmf} = \begin{cases} 1 & \text{with prob } p = \frac{1}{6} \\ 0 & \text{" } \quad \text{" } \quad 1-p \end{cases} \sim \text{Bernoulli}(p)$$

$$C_n = \#\{ i : 1 \le i \le n, \; X_i = 1 \}$$

$$\sim \text{Binomial}(n, p) \quad \Big| \quad P\{C_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$L_k = \underbrace{\#\text{ of throws to get the next win (on } X=1)}_{m}$$

$$L \sim \text{Geometric}(p)$$

$$P\{L = m\} = (1-p)^{m-1}\, p\,,$$

for $m = 1, 2, 3, \dots$

$$E[L] = E\left[\, p \cdot 1 + (1-p)(1+\tilde{L})\right]$$

In other words:

$$L = \begin{cases} 1 & \text{w. prob } p \\ 1+\tilde{L} & \text{w. prob } 1-p \end{cases}$$

$$E[L] = p + (1-p)\big(1 + \underbrace{E[\tilde{L}]}_{E[L]}\big)$$

$$\iff E[L] = p + (1-p) + (1-p)\,E[L]$$

$$\iff p\,E[L] = 1 \qquad \iff E[L] = \frac{1}{p}$$

$$X \sim \text{Bernoulli}(p)$$

$$E[X] = \sum_{x_i} x_i \, P(\{X = x_i\}) = 0 \cdot (1-p) + 1 \cdot p$$

$$= p$$

$$\text{Var}[X] = \underbrace{E[X^2]}_{(0^2 \cdot (1-p) + 1^2 \cdot p)} - \underbrace{(E[X])^2}_{p^2} = p - p^2$$

$$= p(1-p)$$

$$C_n \sim \text{Binomial}(n, p)$$

$$E[C_n] = np$$

$$\boxed{C_n \sim \text{Binom}(100, \tfrac{1}{2})}$$

Ex: Throw a $\underbrace{\text{fair coin}}_{p = \frac{1}{2}}$ $\underbrace{100 \text{ times}}_{n = 100}$

$$P(\text{get } 50 \text{ heads}) = P(\{C_n = 50\})$$

$$= \binom{100}{50} \left(\frac{1}{2}\right)^{50} \left(\frac{1}{2}\right)^{50} = \frac{100!}{50!\,50!} \left(\frac{1}{2}\right)^{100}$$