Finally, to rewrite

$$x \circledast y = \bigwedge_{i,j\in[g]} \bigvee_{k\in[d]} x_k^{(i)} y_k^{(j)}$$

Use de-Morgan & then use R-S with err prob $\frac{1}{4}$

use R-S with err prob $\frac{1}{s} \sim \frac{1}{8g^2}$

$\boxed{s = 8g^2}$

$$\Rightarrow \quad \deg \; O(\log s) = O(\log g)$$

$$\text{err prob} \leq g^2 \cdot \frac{1}{8g^2} + \frac{1}{4} = \frac{3}{8} < \frac{1}{2}$$

$D = \#$ monomials

$$= O\left(g^2 \binom{d}{\log s}\right)^2 = O\left(g^2 \binom{d}{O(\log g)}\right)^2$$

$$\leq O\left(g^2 \; O\left(\frac{d}{\log g}\right)^{O(\log g)}\right)^2$$

$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$

$$\leq O\left(\frac{c \log n}{\log g}\right)^{O(\log g)}$$

$d = c \log n$

Set $g = n^{\frac{1}{t}}$

$\log g = \frac{1}{t}\log n$

$$= O(ct)^{O(\frac{1}{t}\log n)}$$

$$= O(n)^{O(\frac{1}{t}\log(ct))}$$

$\frac{\log t + \log c}{1000 \log c}$

$b \log n$

$\leq O(n^{0.1})$

$a^{b\log n}$
$= n^{b \log a}$

Set $t = 1000 \log c$ $\ll O(n^{0.1})$ $1000 \log e$

$\ll \left(\frac{n}{g}\right)^{0.172}$

$\Rightarrow$ total time $\hat{O}\left(\left(\frac{n}{g}\right)^{\frac{2}{\ }}\right)$

$= \hat{O}\left(n^{2-\frac{2}{t}}\right) = \boxed{O\left(n^{2-\frac{1}{O(\log c)}}\right)}$

APSP

Suffice to solve (min,+) MM of $n \times d$ & $d \times n$ matrixes A, B.

i.e. $k_{ij}^{*} = \arg\min_{k \in [d]} (a_{ik} + b_{kj}) \quad \forall i,j \in [n]$.

Fix $\ell \in [\log d]$.

To compute $f_{ij} = \ell^{th}$ bit of k_{ij}^{*}.

Let $X = \{ k \in [d] : \ell^{th}$ bit of k is $0 \}$.

$\Rightarrow f_{ij} = \bigwedge_{k \in X} \bigvee_{k' \in [d]} [a_{ik'} + b_{k'j} < a_{ik} + b_{kj}]$

$\parallel$ Fredman's trick

$[a_{ik} - a_{ik'} > b_{k'j} - b_{kj}]$

Sort $L = \{ a_{ik} - a_{ik'} : i \in [n],\ k, k' \in [d] \}$

Sort $L = \{ a_{ik} - a_{ik'} : i \in [n],\ k, k' \in [d] \}$
$\cup \{ b_{k'j} - b_{kj} : j \in [n],\ k, k' \in [d] \}$

$$|L| = O(d^2 n)$$

Divide L into d^5 sublists of size $O\left(\frac{n}{d^3}\right)$.

Case 1. $a_{ik} - a_{ik'}$, $b_{k'j} - b_{kj}$ in same sublist for some k, k'

brute force $\Rightarrow O\left(d^2 n \cdot \frac{n}{d^3} \cdot d\right) = O(n^2)$

brute force

Case 2. $a_{ik} - a_{ik'}$, $b_{k'j} - b_{kj}$ in diff sublists for all k, k'

$$f_{ij} = \bigwedge_{k \in X} \bigvee_{k' \in [d]} [a_{ik} - a_{ik'} > b_{k'j} - b_{kj}]$$

$$= \bigwedge_{k \in X} \bigvee_{k' \in [d],\ q \in [d^5]} \underbrace{[a_{ik} - a_{ik'} \text{ is in sublist } q]}_{x^{(i)}_{kk'q}} \cdot \underbrace{[b_{k'j} - b_{kj} \text{ in sublist} < q]}_{y^{(j)}_{kk'q}}$$

AND-of-OR dot products again!

By R-S,

$D = \#\text{monomials} \quad O\left(\underline{d} \cdot \boxed{\binom{d^6}{O(\underline{\log} d)}}^2 \right)$

$\binom{n}{k} \leq n^k$

$$\leq (d^6)^{O(\log d)}$$

$$= d^{O(\log d)}$$

$$= 2^{O(\underline{\log^2 d})}$$

$$\ll n^{0.1}$$

$d = 2^{0.\underline{001}\sqrt{\log n}}$

$2^{\log^2 d} \leq 2^{0.001^2 \log n}$

$\Rightarrow$ (min,+)-MM of $n \times d$ & $d \times n$ in $\tilde{O}(n^2)$ time

$\Rightarrow$ APSP in $\tilde{O}\left(\frac{n}{d} \cdot n^2\right)$

$$= \boxed{\tilde{O}\left(\frac{n^3}{2^{\Theta(\underline{\sqrt{\log n}})}}\right)}$$