for convol3SUM:

search for c_k in $\underbrace{\{ a_i + b_{k-i} : i \text{ in block} \}}_{\text{can sort with no extra real comps (0 cost!)}}$

by binary search

in $O(\log n)$ comps per c_k per block

$\Rightarrow O(n \cdot \frac{n}{d} \cdot \log n)$ total # comps

$\Rightarrow \tilde{O}\left(\frac{n^2}{d}\right)$

$\Rightarrow$ total $\tilde{O}\left(dn + \frac{n^2}{d}\right)$

$\Rightarrow \boxed{\tilde{O}(n^{3/2})}$

Improving Time:

reduces to $\left(\frac{n}{s}\right)^2$ instances of size s

build decision tree of size $2^{\tilde{O}(s^{3/2})} \ll 2^{\delta \log n} \ll n^\delta$.

choose $s \approx \delta \log^{2/3} n$

$\Rightarrow$ total time $O\left(\left(\frac{n}{s}\right)^2 \cdot s^{3/2}\right) = O\left(\frac{n^2}{\sqrt{s}}\right)$

$\approx \boxed{O\left(\frac{n^2}{\log^{1/3} n}\right)}$

3SUM: similar!

$a + b \overset{?}{<} a' + b'$

$a - a' < b' - b$

3SUM: similar.

$a-a' < b'-b$

$a+b+c \overset{?}{<} a'+b'+c'$

Decision Tree Complexity

Rmk: Kane, Lovett, Moran '18

3SUM in $O(n \log^2 n)$ comps

& kSUM " "

& APSP in $O(n^2 \log^2 n)$ comps.

⋮

Time Complexity:

APSP for Reals

Fredman '75 $\sim O(n^3/\log^{1/3} n)$

'90 $\sim O(n^3/\sqrt{\log n})$

'04 $\sim O(n^3/\log^{5/7} n)$

⋮

$\boxed{\text{C'05:} \quad O(n^3/\log n)}$

Han '06: $\sim O(n^3/\log^{5/4} n)$ $\quad 2^{5/4 \log\log n}$

C'07: $\sim O(n^3/\log^2 n)$ $\quad 2^{2 \log\log n}$

Williams '14: $O(n^3/\boxed{c^{\sqrt{\log n}}})$

3SUM for reals

Grönlund-Pettie '14: $\sim O(n^2/\log n)$

C'18: $\sim O(n^2/\log^2 n)$

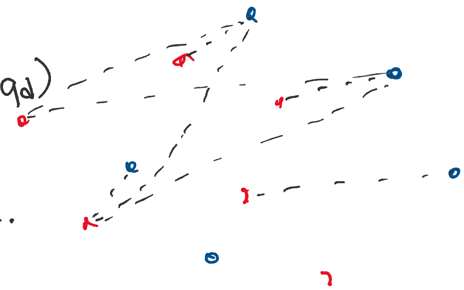
Back to APSP for Reals: C'05

idea - by geometry

Lemma Given $\le n$ red pts & $\le n$ blue pts in $\mathbb{R}^d$, report all K dominating pairs

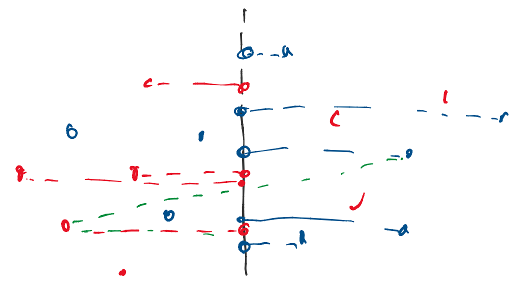
(p,q) p red, q blue, s.t. $p_1 \le q_1, \dots, p_d \le q_d$)

in $O(\underline{n(\log n)^d} + K)$ time.



Pf: divide & conquer

take median-x



$$T_d(n) = 2T_d\left(\frac{n}{2}\right) + T_{d-1}(n) + O(n)$$

$$T_0(n) = O(n) \qquad \text{(ignore } +K)$$

$$\Rightarrow \quad T_1(n) = 2T_1\left(\tfrac{n}{2}\right) + O(n) \Rightarrow T_1(n) = O(n\log n)$$

$$\Rightarrow \quad T_2(n) = 2T_2\left(\tfrac{n}{2}\right) + O(n\log n) \Rightarrow T_2(n) = O(n\log^2 n)$$

$$\vdots$$

$$T_d(n) = O(\underline{n\log^d n}). \qquad \square$$

Slightly better analysis:

$$\binom{n}{k} \le \left(\frac{en}{k}\right)^k$$

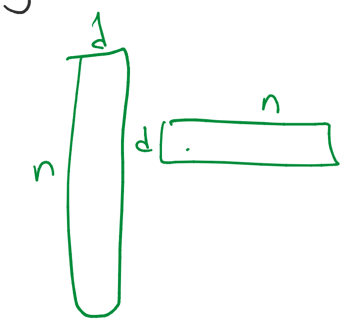
$$O\left(n \binom{\log n + d}{d} + K \right)$$

$$\le O\left(n \left(\frac{e(\log n + d)}{d}\right)^d + K \right).$$

Lemma can compute $(\min,+)$-MM

[illegible] $\times n$ matrix B

Lemma can compute $(\min,+)$-MM of $n \times d$ matrix A & $d \times n$ matrix B in $O(n^2)$ time for $d = \delta \log n$.



Pf: Fix $k_0 \in [d]$.

Subproblem: Find all i,j s.t.

$$\arg\min_{k\in[d]} (a_{ik} + b_{kj}) = k_0.$$

i.e. $a_{ik_0} + b_{k_0 j} \le a_{ik} + b_{kj} \quad \forall k \in [d].$

i.e. $a_{ik_0} - a_{ik} \le b_{kj} - b_{k_0 j} \quad \forall k \in [d]$ by Fredman's trick

i.e. point $(a_{ik_0} - a_{i1}, \ldots, a_{ik_0} - a_{id})$ is dominated by point $(b_{1j} - b_{k_0 j}, \ldots, b_{dj} - b_{k_0 j})$

by Lemma, $O\left(n \left(\frac{e(\log n + d)}{d}\right)^d + \text{output size} \right)$

total time over all $k_0 \in [d]$: $O\left(dn \left(\frac{e(\log n + d)}{d}\right)^d + \cancel{\text{total output size}}\ n^2 \right)$

for $d = \delta \log n$: $O\left(\underbrace{dn \left(\frac{e(1+\delta)\cancel{\log n}}{\delta \cancel{\log n}}\right)^{\delta \log n}} + n^2 \right)$

$$O\left(\underbrace{n \log n \cdot n^{\underbrace{\delta \log \frac{e(1+\delta)}{\delta}}}} + n^2 \right)$$

$\leq \quad O(n^2)$, for suff. small δ.

$\square$

Cor (min, +)-MM of $n \times n$

in $O\left(\frac{n}{d} \cdot n^2\right) = \boxed{O\left(\frac{n^3}{\log n}\right)}$ time

& so APSP in