$$\Rightarrow \text{total time } O\left(\left(\frac{n}{w'}\right)^2\right) \le \boxed{O\left(\frac{n^2}{(\log_\sigma n)^2}\right)}$$

Ex 3: 3SUM for ints

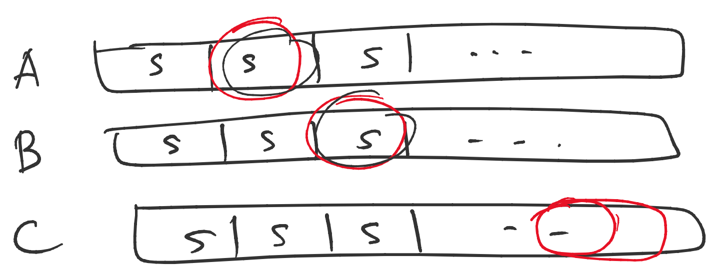
Baran, Demaine, Patrascu '05 $\quad O\left(\frac{n^2}{\log^2 n}(\log\log n)^2\right)$.

idea - hashing again!

$h(x) = x \bmod p$. with rand $p \in (R/2, R]$ for a small R.

Warm-up with Convol-3SUM:

divide into $\frac{n}{s}$ blocks of s numbers



$\Rightarrow$ $\left(\frac{n}{s}\right)^2$ subprobs of $\underbrace{O(s) \text{ numbers}}$

after hashing, $O(s \log R)$ bits

Solve each subproblem by table lookup of size $2^{O(s \log R)} \ll n$

choose $s = \delta \frac{\log n}{\log\log n}$ $\quad R = \log^c n$

$$\Rightarrow O\left(\left(\frac{n}{s}\right)^2\right) = O\left(\frac{n^2}{\log^2 n}(\log\log n)^2\right)$$

for $a_i + b_j \ne c_{i+j}$,

$\Pr[h(a_i + b_j) = h(c_{i+j})] \lesssim \tilde{O}\left(\frac{1}{R}\right)$

for $a_i + b_j \neq c_{i+j}$,

$$\Pr[\,h(a_i + b_j) = h(c_{i+j})\,] \lesssim \tilde{O}\left(\frac{1}{R}\right)$$

$\Rightarrow$ expected # false positives $\lesssim \tilde{O}\left(\frac{n^2}{R}\right)$

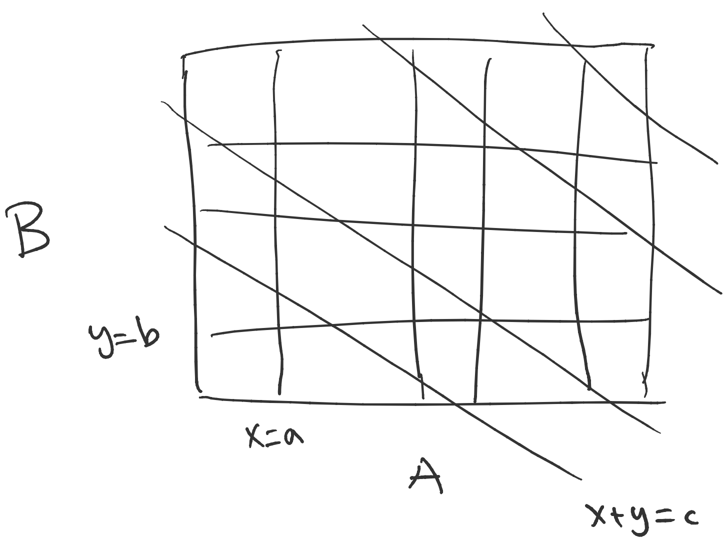
$\Rightarrow$ time for checking these is $\tilde{O}\left(\frac{n^2}{R}\cdot s^2\right)$

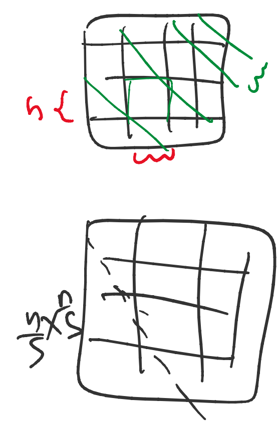
$$\ll O\left(\frac{n^2}{\log^{c-2} n}\right)$$

final bd $\boxed{O\left(\frac{n^2}{\log^2 n}(\log\log n)^2\right)}$.

for 3SUM; sort A, B, C,

then divide into $\frac{n}{s}$ blocks of size s





$O\left(\left(\frac{n}{s}\right)^2\right)$ cells $\Rightarrow$ $O\left(\left(\frac{n}{s}\right)^2\right)$ subprobs of size $O(s)$.

rest is same.

What about reals?

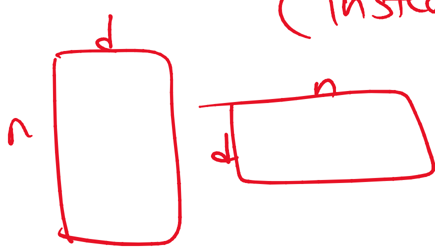
APSP for Reals

first consider decision tree complexity

= # comparisons on real numbers

Fredman '75: APSP can be solved in $O(n^{2.5})$ comps!

Lemma can compute $(\min,+)$-MM of $n \times d$ & $d \times n$ matrix using $\tilde{O}(d^2 n)$ comps (instead of $O(dn^2)$)



Pf: key idea - $a_{ik} + b_{kj} \overset{?}{\leq} a_{ik'} + b_{k'j}$

$$\iff \underbrace{a_{ik} - a_{ik'}}_{} \overset{?}{\leq} b_{k'j} - b_{kj}$$

"Fredman's trick"

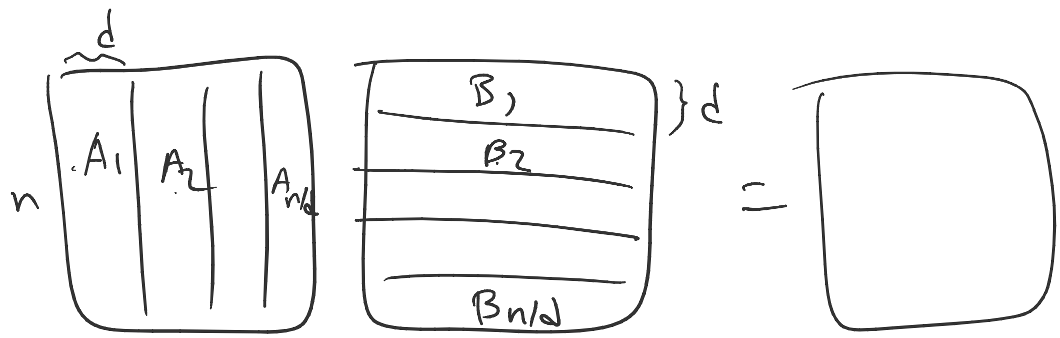
Sort $\{ a_{ik} - a_{ik'} : i \in [n],\ k, k' \in [d] \}$
$\cup \{ b_{k'j} - b_{kj} : j \in [n],\ k, k' \in [d] \}$

$\Rightarrow$ $O(d^2 n)$ numbers
$O(d^2 n \log(d^2 n))$ time

Afterwards, no more real comps needed! $\square$

Thm can compute $(\min,+)$-MM of 2 $n \times n$ real matrices in $\tilde{O}(n^{2.5})$ comps

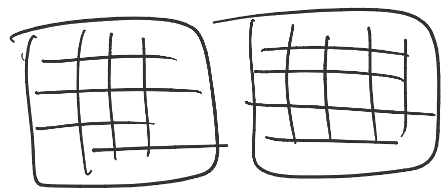
Pf: reduce to $\frac{n}{d}$ products of $n \times d$ & $d \times n$



$$\Rightarrow \tilde{O}\left(\frac{n}{d} \cdot d^2 n + \frac{n}{d} \cdot n^2\right)$$
$$= \tilde{O}\left(d n^2 + \frac{n^3}{d}\right)$$

Choose $d = \sqrt{n}$. $\square$

Improving Time:



reduce to $\left(\frac{n}{s}\right)^3$ $(\min,+)$-MM of $s \times s$ matrices

build decision tree of size/time $2^{\tilde{O}(s^{5/2})} \ll n$

Choose $s \approx \delta \log^{2/5} n$

Choose $s \approx \delta \log^{2/5} n$

$$\Rightarrow \sim O\left(\left(\frac{n}{s}\right)^3 \cdot s^{2.5}\right) = O\left(\frac{n^3}{\sqrt{s}}\right)$$

$$\approx \boxed{O\left(\frac{n^3}{\log^{1/5} n}\right)}$$

3SUM for Reals

40 yrs later!

Grönlund - Pettie '14: 3SUM can be solved in $O(n^{1.5})$ comps

$a+b < a'+b'$

Warm-up with (min,+)-Convol or Convol3SUM:

[BCDEHILPT'06]

divide $A = | A_1 | A_2 | \ldots$ $\quad \frac{n}{d}$ blocks (blocks of size d)

$B = | B_1 | B_2 | \ldots$

$C = | C_1 | C_2 | \ldots$

idea - Fredman's trick

$$a_i + b_{k-i} \overset{?}{\leq} a_{i'} + b_{k-i'}$$

$$\Longleftrightarrow \quad a_i - a_{i'} \overset{?}{\leq} b_{k-i'} - b_{k-i}$$

$$\iff a_i - a_{i'} \overset{?}{\lesssim} b_{k-i'} - b_{k-i}$$

Preproc: Sort $\{ a_i - a_{i'} : i, i' \text{ in same block} \}$
$\cup \{ b_j - b_{j'} : j, j' \text{ in same block} \}$

in $\tilde{O}\left(\frac{n}{d} \cdot d^2\right) = \tilde{O}(dn)$ time

Afterwards, can compute $\underset{i}{\min}\,(a_i + b_{k-i})$ in each block with no real comp.

for (min,+)-convol:

$\Rightarrow$ can compute c_k in $O\left(\frac{n}{d}\right)$ comps.

$\Rightarrow$ total $\tilde{O}\left(dn + \frac{n^2}{d}\right)$

Choose $d = \sqrt{n}$ $\Rightarrow$ $\boxed{\tilde{O}(n^{3/2})}$.

for convol3SUM, to be cont'd ...