More Cond. Lower Bds from 3SUM

\((\text{3SUM})\) Triangle in \(O(n^{3.5})\) \(\uparrow\) \(\text{APSP in } O(n^{3.8})\)

Rank: can reduce zero-lit triangle back to 3SUM but with much weaker bds

Given weighted triangle graph \(G_f(v,E)\),

\[
\begin{align*}
wx &= yz \\
\text{cost number } w(xy) &= xM - yM^2 \\
\text{if } xy \in E, \quad \text{w(yz) } &= yM^2 - zM^3 \\
\text{if } xz \in E, \quad \text{w(yz) } &= w(xy) + 2M^3 - xM \\
\end{align*}
\]

If 3SUM could be solved in \(T(n)\) time,

Zero-lit triangle could be solved in \(O\left(\frac{T(n)}{n^2} \right)\) time

Triangle Listing Problem

Given unweighted graph \(G=(v,E)\) with \(n\) edges,

report all \(K\) triangles

Recall: Alon, Yuster, Zwick '97

\(O\left(\frac{2^\omega n^{2.5}}{\omega^3} \right) \leq O(m^{1.91})\)

(Bjorklund et al. '14)

Same bound for \(K \leq m\).

Thin (Patrascu '10)

Assuming \(\text{3SUM} \Rightarrow \text{APSP}\)

no \(O(m^{4/3 - \epsilon})\) algm for triangle listing

for \(K = m\).

(lots of applns to data structure lower bds ...)

Problem: Set Intersection Queries

Given \(N\) sets \(S_1, \ldots, S_N\),

build data structure to answer queries:

\(\text{report all elements in } S_i \cap S_j\)
Given \( N \) sets \( S_1, \ldots, S_N \), build data structure to answer queries:

- given \( i, j \), report all elems in \( S_i \cap S_j \).

(in offline case, all queries are given in advance)

\[
\begin{align*}
\text{Reduce } \text{3SUM} & \rightarrow \text{Offline Set Intersect. Queries} \\
\end{align*}
\]

To solve Conv3Sum for \( a_0, \ldots, a_{n-1} \), decide \( i, j \):

\[ a_i + a_j = a_m \]

Use hash fn \( h(x) = x \mod t \) for rand \( t \in (R/2, R) \)

Define bucket \( B_k = \{ j : h(a_j) = k \} \) \( k \in [R] \)

(call bucket \( B_k \) good if \( |B_k| \geq \mathcal{O}(\sqrt{R}) \)
(Prob [ans in 3 good buckets] \( \approx \frac{1}{3} \chi^2(n) \)).

1. for \( i \in [n] \) do
   2. for \( l \in [R] \) do
   3. find all \( j \) s.t. \( h(a_j) = l \)
      and \( h(a_{ij}) = l + h(a_i) \)
      or \( l + h(a_i) - l \).
   4. for each such \( j \), if \( a_i + a_j = a_{ij} \), exit & return yes
5. return no

Def: \( S+x = \{ s+x : s \in S \} \) "shifted set"

Line 3 reduces intersecting \( B_k \) and \( B_{k+tw(a_i)}-i \)
but \# possible sets = \( O(Rn) \) too big

Idea - write \( i = xd + y \), \( x \in \llbracket n/d \rrbracket \), \( y \in \llbracket d \rrbracket \)

reduces to intersecting \( B_k+y \) and \( B_{k+tw(a_i)}-xd \)

\# possible sets \( \approx d \times \frac{n}{d} \)

\# possible sets
Choose \( d = \sqrt{n} \)

\[
N = \# \text{ sets} = O(R\sqrt{n})
\]

\[
\text{total set size} = \tilde{O}(R\sqrt{n} \cdot \frac{n}{R}) = \tilde{O}(n^{3/2})
\]

\[
\# \text{ queries} = O(nR) = \tilde{O}(nR)
\]

Expected total output size of queries

\[
= \text{expected number of false positives} + 1
\]

\[
= \# (i,j) \text{ with } a_i + a_j \neq a_{ij} \text{ but } h(a_i + a_j) = h(a_{ij})
\]

\[
\leq \tilde{O}\left(n^2 \cdot \frac{1}{R}\right)
\]

Choose \( R = \sqrt{n} \) \( \Rightarrow \)

\[
\text{total set size} = \tilde{O}(n^{3/2})
\]

\[
\# \text{ queries} = \tilde{O}(n^{3/2})
\]

\[
\text{total output size} = \tilde{O}(n^{3/2})
\]

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**Reduce Offline Set Intersect. Queries \( \rightarrow \) Triangle Listing**

Given sets \( S_1, \ldots, S_n \), & queries,

Create tripartite graph

\[
\begin{align*}
m &= \# \text{ edges} = \text{total set size} + \text{total } \# \text{ queries} \\
    &= \tilde{O}(n^{3/2})
\end{align*}
\]

\[
N = \# \text{ triangles} = \text{total output size} = \tilde{O}(n^{3/2})
\]
\[ K = \# \text{triangles} = \text{total output size} = \tilde{O}(n^{3/2}) \]

\[ \implies \text{time} \tilde{O}\left( T_{\text{tri-list}}(\tilde{O}(n^{3/2})) \right) \leq \tilde{O}\left( (n^{3/2})^{4/3 - \delta'} \right) \]

\[ = \tilde{O}\left( n^{2 - \delta'} \right). \]

\textbf{Remark:} Vassilevska W. & Xu '20:

- Same lower bd for triangle listing
- under \#PSP Conj

CUX '22:

- realAPSP Conj.