October 5, 2022 10:38 AM

Cond. LBs from (min,+)-Convolution

Problem Given $a_0, \dots, a_{n-1}$, $b_0, \dots, b_{n-1}$,
Compute $c_k = \min_i (a_i + b_{k-i})$

Conjecture No $O(n^{2-\delta})$ alg'm for (min,+)-Convol.

stronger than APSP Conj.

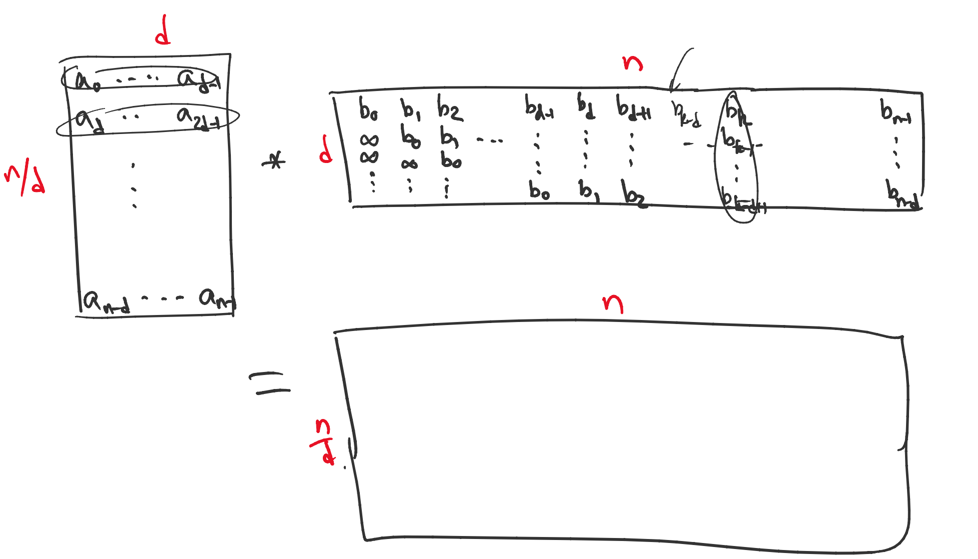
Thm If APSP has $O(n^{3-\delta})$ alg'm for some $\delta > 0$,
then (min,+)-convol has $O(n^{2-\delta'})$ alg'm for some $\delta' > 0$.

(BCDEHILPT'14)

Pf: Reduce (min,+)-convol $\to$ (min,+)-MM

Suppose (min,+)-MM could be solved in $M^*(n) = O(n^{3-\delta})$ time

To solve (min,+)-Convol:
given a_i's, b_i's.



each c_k can be recovered from $\frac{n}{d}$ output entries

Total time $O\left(M^*\left(\frac{n}{d}, d, n\right) + \cancel{n \cdot \frac{n}{d}}\right)$

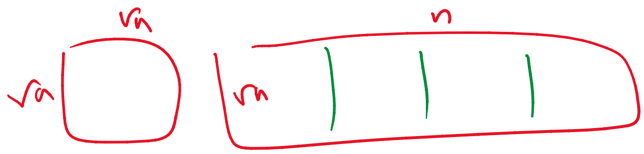
time for rect (min,+)-MM

Choose $d = \sqrt{n}$: $O(M^*(\sqrt{n}, \sqrt{n}, n))$

$= O(\sqrt{n}\; M^*(\sqrt{n}))$

Choose $d = \sqrt{n}$: $O(M(\ldots))$

$$\le O\left(\sqrt{n}\; M^*(\sqrt{n})\right)$$



$$\le O\left(\sqrt{n} \cdot (\sqrt{n})^{3-\delta}\right)$$

$$= O\left(n^{2-\delta/2}\right).$$

$\square$

Problem: 0/1 Knapsack

Given n items (w_i, p_i), $i = 1, \ldots, n$, and T,

weight ↗ ↖ profit ↗ capacity (all integers).

find subset $I \subseteq \{1, \ldots, n\}$, maximizing $\sum_{i \in I} p_i$ s.t. $\sum_{i \in I} w_i \le T$.

(subset sum is special case $w_i = p_i$)

Unbounded Knapsack: similar but I is a multiset

Standard DP: $O(nT)$ time

Is there an $\tilde{O}(T)$ alg'm for knapsack, like Bringmann?

Thm (Cygan et al. '17 / Künnemann et al. '17)

$(\min,+)$-Convol has $O(n^{2-\delta})$ alg'm for ints.

$\Leftrightarrow$ 0/1 Knapsack has $O(T^{2-\delta'})$ alg'm

$\Leftrightarrow$ Unbdd Knapsack " $O(T^{2-\delta''})$ "

Pf: $(\Rightarrow)$ Unbdd Knapsack reduces to $(\min,+)$-Convol by repeated squaring:

Pf: ($\Rightarrow$) Unbdd Knapsack reduces to ...
by repeated squaring:

let $f^{(\ell)}(j) =$ max profit with capacity j using $\le \ell$ items

$$f^{(\ell)}(j) = \max_{j'} \underbrace{\left(f^{(\ell/2)}(j') + f^{(\ell/2)}(j-j') \right)}_{(\max,+)\text{-Convol}}$$

0/1 Knapsack reduces to $(\min,+)$-Convol by modifying Bringmann's alg. ...,

($\Leftarrow$) Will reduce $(\cancel{\min}\ \max,+)$-Convol to Knapsack ...

Step 1. Reduce $(\max,+)$-Convol to $(\max,+)$-Convol-Decis

given 3 seqs $a_0, \ldots, a_{n-1}$, $b_0, \ldots, b_{n-1}$, $c_0, \ldots, c_{n-1}$
for each k, decide $\exists i$, $a_i + b_{k-i} > c_k$.

by binary search

Step 2. Reduce $(\max,+)$-Convol-Decis $\to$ Report-One $(\max,+)$-Convol-Decis

decide $\exists k, \exists i$, $a_i + b_{k-i} > c_k$

by same idea as All-Edge NWT $\to$ Report-One NWT

$\forall k, i$,
$a_i + b_{k-i} \le c_k$.

$\Updownarrow$

$\forall i, j$,
$a_i + b_j \le c_{i+j}$.
$+Mi \quad +Mj \quad +M(i+j)$

Problem: Superadditivity

given one sequence $f_0, \ldots, f_{n-1} \ge 0$,
decide $\forall i, j$. $f_i + f_j \le f_{i+j}$

$a_{i+1} + M(i+1)$
$> a_i + Mi$
$a_i - a_{i+1} < M$

Step 3 Reduce Report-One $(\max,+)$-Convol-Decis

$a_i - a_{i+1} < M$

Step 3. Reduce Report-One $(\max,+)$-Convol-Decis $\longrightarrow$ Superadditivity

Given $a_0, \ldots, a_{n-1}$, $b_0, \ldots, b_{n-1}$, $c_0, \ldots, c_{n-1}$

w.l.o.g., assume a_i's, b_i's, c_i's are increas & are in $[U]$

(if not, add linear fn Mi for large M)

Combine into one sequence:

$\forall i \in [n]$,

$$f_i = 0$$
$$f_{n+i} = U + a_i$$
$$f_{2n+i} = 4U + b_i$$
$$f_{3n+i} = 5U + c_i.$$

Step 4. Superadditivity $\longrightarrow$ Unbdd Knapsack

To solve Superadditivity, given $f_0, \ldots, f_{n-1}$:

for each $i \in [n]$,

create items.

type 1: (i, f_i) — weight, profit

type 2: $(T-i, U-f_i)$

with $T = 2n$.