September 16, 2022 10:48 AM

Appl'ns of Matrix Mult. (Cont'd)

Appl'n 2: Transitive Closure

$n = |V|$, $m = |E|$, $m \gtrsim n$

Given dir. graph $G = (V, E)$,
$\forall s, t \in V$, decide $\exists$ path $s \rightsquigarrow t$

Standard alg'ms:

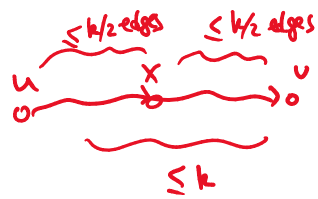
n DFS/BFSs: $O(mn) \lesssim O(n^3)$

DP (Warshall): $O(n^3)$

Alg'm 1: repeated squaring

let $c_{uv}^{(k)}$ = true iff $\exists$ path $u \rightsquigarrow v$ with $\le k$ edges ($\le k-1$ hops)

$$c_{uv}^{(k)} = \bigvee_{x \in V} \left(c_{ux}^{(k/2)} \wedge c_{xv}^{(k/2)} \right)$$

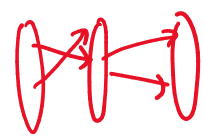


Boolean MM
$$C^{(k)} = C^{(k/2)} * C^{(k/2)}$$

for $k = 1, 2, 4, \ldots, n$

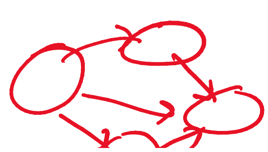
...

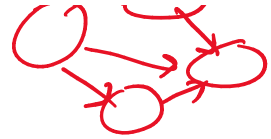
$$\Rightarrow \boxed{O(n^{\omega} \log n)} \le O(n^{2.373})$$



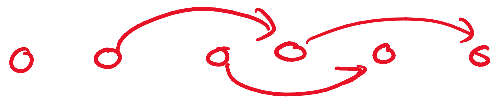
Alg'm 2: (Munro '71)

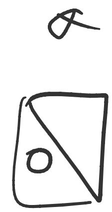
May assume that G is a DAG.
(by strongly connected comps & meta-graph)





Topologically sort vertices
i.e. all edges are of the form (i,j) with $i<j$.





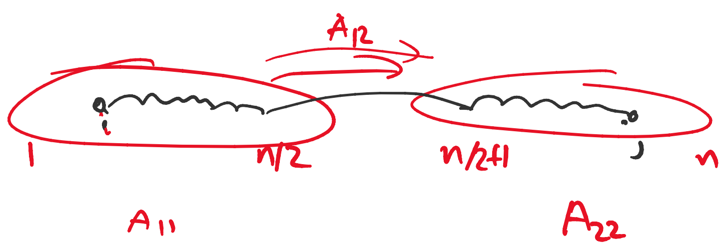
Let $a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{else} \end{cases}$

To compute $a^*_{ij} = \begin{cases} 1 & \text{if } \exists \text{ path } i \leadsto j \\ 0 & \text{else} \end{cases}$

idea: D & C

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$

Then $A^* = \begin{pmatrix} A^*_{11} & A^*_{11} A_{12} A^*_{22} \\ 0 & A^*_{22} \end{pmatrix}$



$\Rightarrow T(n) = 2T(n/2) + O(n^\omega)$

$\Rightarrow \boxed{O(n^\omega)}$

$2\left(\frac{n}{2}\right)^\omega + 4\left(\frac{n}{4}\right)^\omega + 8\left(\frac{n}{8}\right)^\omega + \dots$

$= \frac{n^\omega}{2^{\omega-1}} + \frac{n^\omega}{4^{\omega-1}} + \dots$

$= O(n^\omega)$

Appl'n 3: All-Pairs Shortest Paths (APSP)

Given $G=(V,E)$, with edge weights,

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$\forall s,t \in V$, compute $d(s,t)$ = shortest path dist from s to t.

Known: n Dijkstra $\Rightarrow$ $\tilde{O}(mn)$
Floyd-Warshall $\Rightarrow$ $O(n^3)$

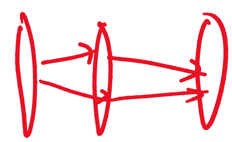
Repeated squaring

$$c_{uv}^{(k)} = \min_{x \in V} \left(c_{ux}^{(k/2)} + c_{xv}^{(k/2)} \right)$$

$\Rightarrow$ $(\min,+)$ - matrix mult.

believe to require near-n^3 time.

Munro generalizes also



Will now consider unweighted case
or small integer weights in $[c]$

Zwick's Alg'm ('02)

Lemma Given $n \times n$ matrices A, B where all entries are in $[c] \cup \{\infty\}$
can compute $(\min,+)$-MM in $\tilde{O}(c\, n^{\omega})$ time

Pf: Define new matrices A', B':

$$a_{ij}' = M^{a_{ij}} \qquad b_{ij}' = M^{b_{ij}} \quad \text{for large } M$$

Compute standard MM $C' = A'B'$:

$$c_{ij}' = \sum_{k=1}^{n} a_{ik}' b_{kj}' = \sum_{k=1}^{n} M^{a_{ik}+b_{kj}}$$

most signif. digit position $\qquad \max_k (a_{ik}+b_{kj})$

Choose $M = n+1$.

Runtime: $O(n^\omega)$ arith ops on numbers in $[M^c]$ ← $O(c \log M)$ bits

each arith op takes $\tilde{O}(c \log M) = \tilde{O}(c)$ time by FFT.

$\Rightarrow$ total $\tilde{O}(c n^\omega)$. $\square$

issue - when reducing APSP to $(\min,+)$-MM, entry values get bigger $\rightarrow [c]$

$\Rightarrow \tilde{O}(c n^{\omega+1})$ bad!

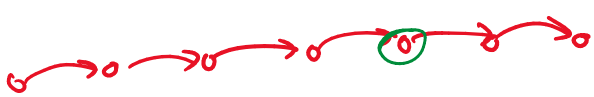
first idea - short vs. long paths

Case 1. Short paths with $\le L$ edges/hops.

repeated squaring to $O(\log n)$ $(\min,+)$-MM in $[cL]$

$\Rightarrow \tilde{O}(cLn^\omega)$

Case 2. long paths with $\ge L$ edges/hops.



Hitting Set Lemma $\quad \exists$ subset $R \subseteq V$ of size $O(\frac{n}{L} \log n)$

that hits all shortest paths with $\geq L$ edges.