CS 598TMC, Fall 2022
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Homework 3 (due Nov 3 Thursday 5pm)

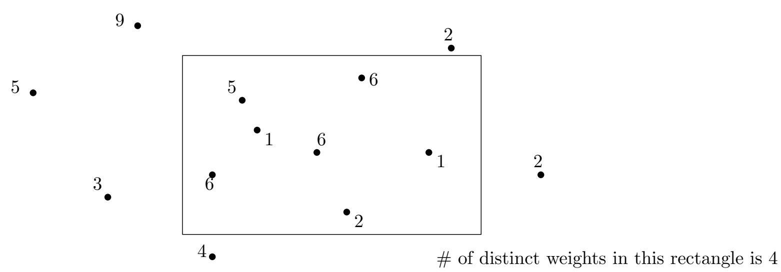
Instructions: see previous homework.

1. [*30 pts*] In class, we have given a reduction from max-plus convolution to the "report-one decision" version of the problem: given sequences $a_0, \ldots, a_{n-1}$, $b_0, \ldots, b_{n-1}$, $c_0, \ldots, c_{n-1}$ (all integers in $[U]$), decide whether there exist k and i with $a_i + b_{k-i} > c_k$.

   Give a further reduction from the report-one decision problem to the following equality variant of the problem: given sequences $a_0, \ldots, a_{n-1}$, $b_0, \ldots, b_{n-1}$, $c_0, \ldots, c_{n-1}$ (all integers in $[U]$), decide whether there exist k and i with $a_i + b_{k-i} = c_k$. Thus, this variant does not have an $O(n^{2-\delta})$-time algorithm, assuming the (integer) max/min-plus convolution conjecture.

   [Hint: this should be an easy modification of a reduction we have seen from class, about APSP/min-plus matrix multiplication.]

2. [*34 pts*] Consider the following *point-rectangle-counting* problem: We are given a set P of N points in 2D, where each p has a weight $w(p)$. We are also given N axis-aligned rectangles $R_1, \ldots, R_N$. For each rectangle R_k, we want to count the number of *distinct* weights among the points of P that are inside R_k. (The output thus consists of N numbers.)

   (Note: if the problem is changed to counting the number of points of P inside R_k, $O(N \log N)$-time algorithms were known; but counting the number of distinct weights seems more difficult...)

   Give a reduction of matrix multiplication with matrix entries in $\{0, 1\}$ to the point-rectangle-counting problem, to prove an $\Omega(N^{\omega^*/2})$ lower bound for the latter problem, where ω^* is the exponent for 0/1-matrix multiplication. Thus, if $\omega^* > 2$, this would rule out near linear time algorithms for the point-rectangle-counting problem.

   [Hint: to multiply two $n \times n$ matrices, create a set of $N = O(n^2)$ points, all lying on two diagonal lines, with $O(n)$ groups of $O(n)$ points...]

3. [*36 pts*] Although it is open whether the conjectures about SETH/OV and APSP/min-plus matrix multiplication are related, in this question you will relate OV to a weird variant of matrix multiplication.

Given two $n \times n$ matrices $A = (a_{ij})_{i,j\in[n]}$ and $B = (b_{ij})_{i,j\in[n]}$, in the *weird matrix multiplication problem* (which could be called "distinct-equality" matrix multiplication), we want to compute the matrix $C = (c_{ij})_{i,j\in[n]}$, denoted by $C = A \odot B$, where

$$c_{ij} = |\{a_{ik} : \ a_{ik} = b_{kj}\}|.$$

In other words, c_{ij} is the number of distinct elements x such that $x = a_{ik} = b_{kj}$ for some $k \in [n]$.

To understand this problem better, we make another definition: given two n-dimensional vectors $\mathbf{u} = (u_0, \ldots, u_{n-1})$ and $\mathbf{v} = (v_0, \ldots, v_{n-1})$, define the *weird dot product* to be

$$\mathbf{u} \odot \mathbf{v} = |\{u_k : \ u_k = v_k\}|.$$

(For example, $(5, 7, 4, 3, 7) \odot (0, 7, 6, 3, 7)$ is 2, since $|\{7, 3, 7\}| = 2$.) Thus, the weird matrix product of two matrices A and B consists of the weird dot products between all row vectors of A and all column vectors of B.

(a) [*12 pts*] Let d and g be parameters. Given a dg-dimensional Boolean vector $\mathbf{u} = (u_0, \ldots, u_{dg-1})$, define a new dg-dimensional vector $\varphi(\mathbf{u}) = (u'_0, \ldots, u'_{dg-1})$, where for each $\ell \in [g]$ and $k \in [d]$,

$$u'_{\ell d+k} = \begin{cases} \ell & \text{if } u_{\ell d+k} = 1 \\ g+1 & \text{if } u_{\ell d+k} = 0 \end{cases}$$

Given a d-dimensional Boolean vector $\mathbf{v} = (v_0, \ldots, v_{d-1})$, define a new dg-dimensional vector $\psi(\mathbf{v}) = (v'_0, \ldots, v'_{dg-1})$, where for each $\ell \in [g]$ and $k \in [d]$,

$$v'_{\ell d+k} = \begin{cases} \ell & \text{if } v_k = 1 \\ g+2 & \text{if } v_k = 0 \end{cases}$$

Prove that the weird dot product between $\varphi(\mathbf{u})$ and $\psi(\mathbf{v})$ is strictly less than g if and only if at least one of the g vectors $(u_0, \ldots, u_{d-1})$, $(u_d, \ldots, u_{2d-1})$, $\ldots$, $(u_{d(g-1)}, \ldots, u_{dg-1})$ is orthogonal to $\mathbf{v}$ (i.e., has zero standard dot product with $\mathbf{v}$).

(b) [*24 pts*] Give a reduction from OV to weird matrix multiplication, proving that there is no $O(n^{3-\delta})$-time algorithm for weird matrix multiplication, assuming the OV conjecture.

[Hint: given two sets of N vectors in d dimensions, divide the first set into N/g groups of g vectors each, and apply the $\varphi(\cdot)$ mapping from (a) to each group. . .]