Non-orthogonal Range Searching

triangle range search (or simplex)
halfplane range search (or halfspace)

History

⇒ Willard '82
Edelsbrunner–Welzl '86
Hausler–Welzl '88
Welzl '88
Chatéelle–Sharir–Welzl '90
⇒ Matoušek '91
Matoušek '92
C'10
⇒ Clarkson '87
⇒ Matoušek '92

Method 1: Willard '82

Ham-Sandwich Cut Theorem

Given any 2 point sets P, Q in \( \mathbb{R}^2 \),

∃ a line that simultaneously bisects P & Q.
Given direction \( v \), let \( l_p(v) \) be line bisecting \( P \) pointing in dir. \( v \)
\[
l_q(v) = \text{ "\( \quad \) Q "\( \quad \) "}
\]

Before \( v = (1,0) \)

Say \( l_p \) left of \( l_q \)

As \( v \) rotates, \( l_p \) & \( l_q \) move continuously

After \( v = (-1,0) \)

\( l_p \) right of \( l_q \)

By intermediate value thm,
\[
\exists v, \text{ s.t. } l_p(v) = l_q(v). \quad \square
\]

(Megiddo '81 : \( O(n) \) time)

(in \( \mathbb{R}^d \), need Borsuk-Ulam Thm)

Cor

Given any set \( P \) of \( n \) pts in \( \mathbb{R}^2 \)
\[
\exists 2 \text{ lines which partition } P \text{ into 4 subsets of } \frac{n}{4} \text{ pts}
\]
2. Ham-sandwich cut

1. Pick median vertical line

Recursion $\Rightarrow$ Partition tree

Space $\mathcal{O}(n)$

Preproc time $\Rightarrow \mathcal{O}(n \log n)$

\[
P(n) = 4 \cdot P\left(\frac{n}{4}\right) + \mathcal{O}(n)
\]

Query: given halfplane $q$,

- Recurse in cells crossed by line $2q$.
- 3 out of the 4

\[
Q(n) = 3 \cdot Q\left(\frac{n}{4}\right) + \mathcal{O}(1)
\]

\[
\Rightarrow \mathcal{O}(n^{\log_4 3}) = \mathcal{O}(n^{0.793})
\]

Remark - 8-sectioning in 3D
but no 16-sectioning in 4D!

- Divide into $>4$ parts in 2D?
Method 2:

**Def** (Duality) Given $p = (a, b)$ in 2D, define its dual line $p^*$: $y = ax - b$.

Given line $l: y = ax - eta$, define its dual point $l^* = (a, eta)$.

**Fact**

(i) $p$ above $l$ $\iff$ $l^*$ above $p^*$

(ii) $p$ is on $l$ $\iff$ $l^*$ is on $p^*$

(iii) $l$ is line than $p$, $q$ $\iff$ $l^*$ is intersection of $p^*$, $q^*$

(iv) Given $n$ pts, count pts above query line $\iff$

Given $n$ lines, count # lines below query pt