Near-Linear ε-Emulators for Planar Graphs

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Graph Sparsifier

Given a graph G, a **sparsifier** H is an object that preserves information in G

- e.g. cut, flow, **distance**, etc.

There are many types of distance sparsifiers:

H is a subgraph of G Spanner

H is a minor of G Distance-preserving minor

H is a data structure Distance oracle

Emulator / Planar Emulator

Given a graph G and a terminal set T, an **emulator** is a graph H that contains T, such that for $u, v \in T$,

$$d_G(u, v) \leq d_H(u, v) \leq \alpha \cdot d_G(u, v).$$

We want simultaneously $\alpha \rightsquigarrow 1$ and $|H| \rightsquigarrow k$

We will assume G is planar and require that H is planar

Exercise 1

question Prove that there is always a distance-preserving minor with distortion 1 and size $O(k^4)$

Previous Results

Distortion	Size (lower/upper)		Minor?	
1	$\Omega(k^2)$		no	Chang-Ophelder '20
1		$O(k^4)$	yes	Krauthgamer-Nguyen-Zondiner '14
$1+\varepsilon$	$\Omega(k/\varepsilon)$		yes	Krauthgamer-Nguyen-Zondiner '14
$1+\varepsilon$		$\tilde{O}((k/\varepsilon)^2)$	yes	Cheung-Goranci-Henzinger '16
$1+\varepsilon$		$\tilde{O}(k/\operatorname{poly}\varepsilon)$	no	Chang-Krauthgamer-Tan '22
$O(\log k)$		k	yes	Filster '18

Ideas

separator	Basic tool for any planar graph problem
ε-cover	Basic tool for approximate distance problems
spread	Classic idea in computational geometry; how does it arise in planar graphs?

Separator

Separator Theorem

definition Given a graph G, a separator is a vertex set X, such that $G - X$ is disconnected

Separator Theorem [Lipton-Tarjan '79]

Every planar graph G with size n admits a separator X, such that

- $|X| = O(\sqrt{n})$
- each component of $G - X$ has size at most $2n/3$

Variants of Separator Theorem

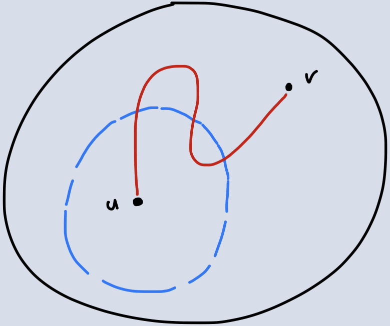
Frederickson '86 (r-division) Divide G into $O(n/r)$ regions with size r, each region has boundary size $O(\sqrt{r})$

Miller '86 X must be a cycle in G

Thorup '01 X must consist of 3 shortest paths, but could be $O(n)$ size

All computable in $O(n)$ time

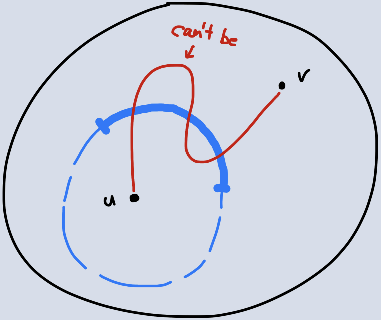
Illustration



ε-Cover

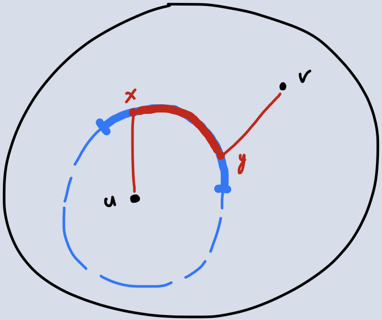
Preserving Distances via Shortest Paths

Illustration



Preserving Distances via Shortest Paths

Illustration



ε-Cover

Given a shortest path P and a vertex $u \notin P$, a vertex set $C(u, P) \in P$ is an **ε-cover** of u on P if

$$\min_{x \in C(u,P)} d(u, x) \leq (1 + \varepsilon) \min_{w \in P} d(u, w)$$

Lemma [Thorup '01]

For every shortest path P and vertex $u \notin P$, there is an ε-cover of u on P with size $O(1/\varepsilon)$, and it can be found in linear time

Exercise 2

question Prove that there is always a $(1+\varepsilon)$-emulator with size $\tilde{O}((k/\varepsilon)^2)$
[Cheung-Goranci-Henzinger '16]

Near-Linear ε-Emulator

Special Case

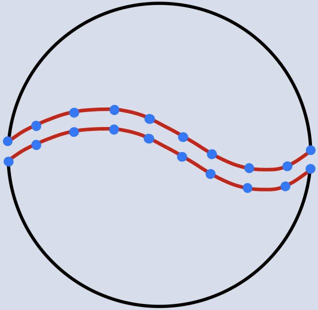
Some simplifying assumptions:

- All terminals lie on the outer face
- No terminal is a cut vertex

This is where the main technical contribution happens

Exponential Portals

Illustration



How to choose the portals?

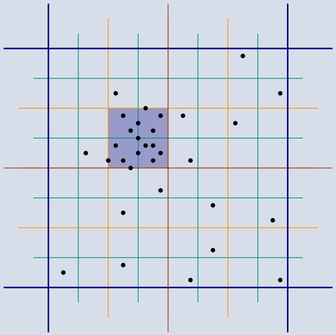
- Exponential portals: $e^{\varepsilon}, e^{2\varepsilon}, e^{3\varepsilon}, \ldots$

Number of portals: $\log \Phi = \log\left(\frac{\max_{u,v\in T} d(u,v)}{\min_{u,v\in T} d(u,v)}\right)$

Spread Reduction

Analogy with Point Sets

Illustration



Hierarchical Partitioning of Terminals

Set a distance parameter μ

If two terminals u, v has $d(u, v) \leq \mu^i$, they belong to the same level-i cluster

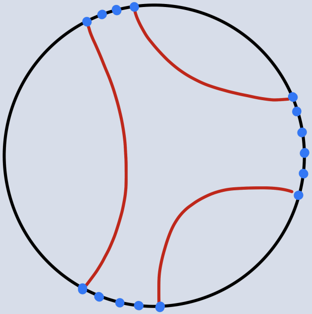
covering At level i, each u belongs to a cluster with diameter $\leq k \cdot \mu^i$

packing If u, v belong to different level-i clusters, then $d(u, v) > \mu^i$

What does a cluster look like?

Beautiful Shape of a Cluster

Illustration



Cousins Are Nice

For a cluster S, hat can we say about its parent's siblings?

- They are $\geq \mu^{i+1}$ away from S
- But S has diameter $\leq k\mu^i$
- Their distances to ∂S are easy to approximate

The New Problem

We want to find a "centroid" cluster S with very small siblings

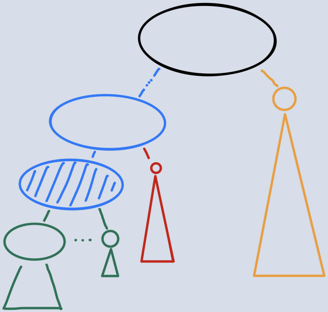
- Parent's siblings don't create portals
- Handle siblings using Thorup's ε-covers

There are many clusters with very small siblings (because large spread)

But what if none of them is a centroid?

The New Plan

Illustration

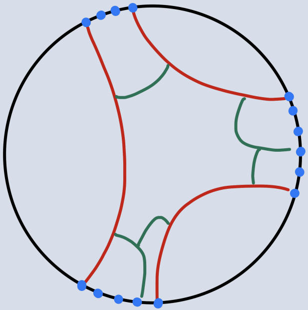


1. Find lowest heavy set with very small siblings
2. The blue cluster can be decomposed into
   - small clusters with very small siblings
   - singletons

all close to blue in the tree

The Final Picture

Illustration



Partitioning of blue looks like this

Now remains to handle the center region

- All terminals are portals we created
- Can artificially increase distances

Summary

Main Ideas

separator	Basic tool for any planar graph problem
ε-cover	Basic tool for approximate distance problems
spread	Novel use of hierarchical partitioning for planar emulator

Final Result

Distortion	Size (lower/upper)		Minor?	
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