PPAD
(Polynomial Parity Argument for Directed graphs)

Menu

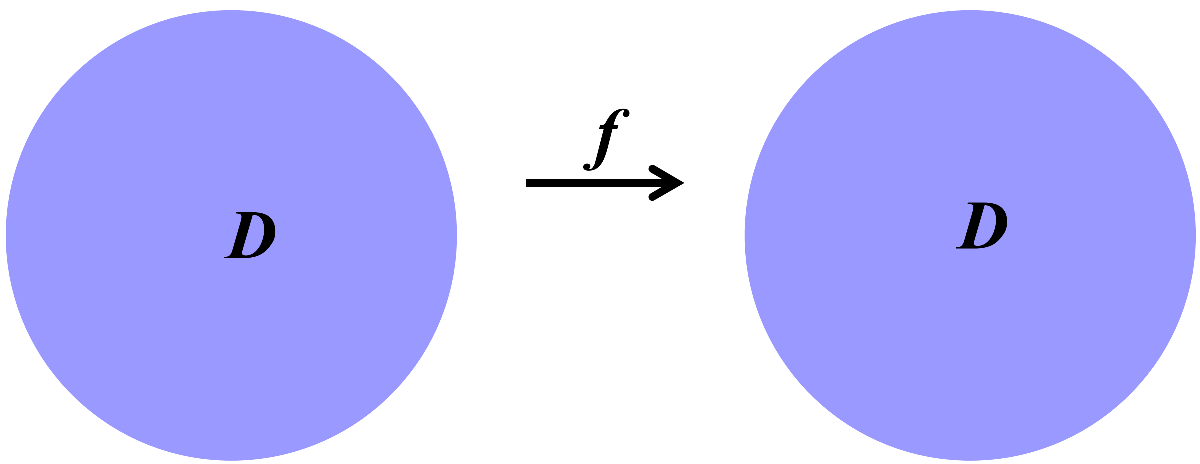
↳ **PPAD- Complete Existence Theorems: Brouwer, Sperner**

Brouwer's Fixed Point Theorem

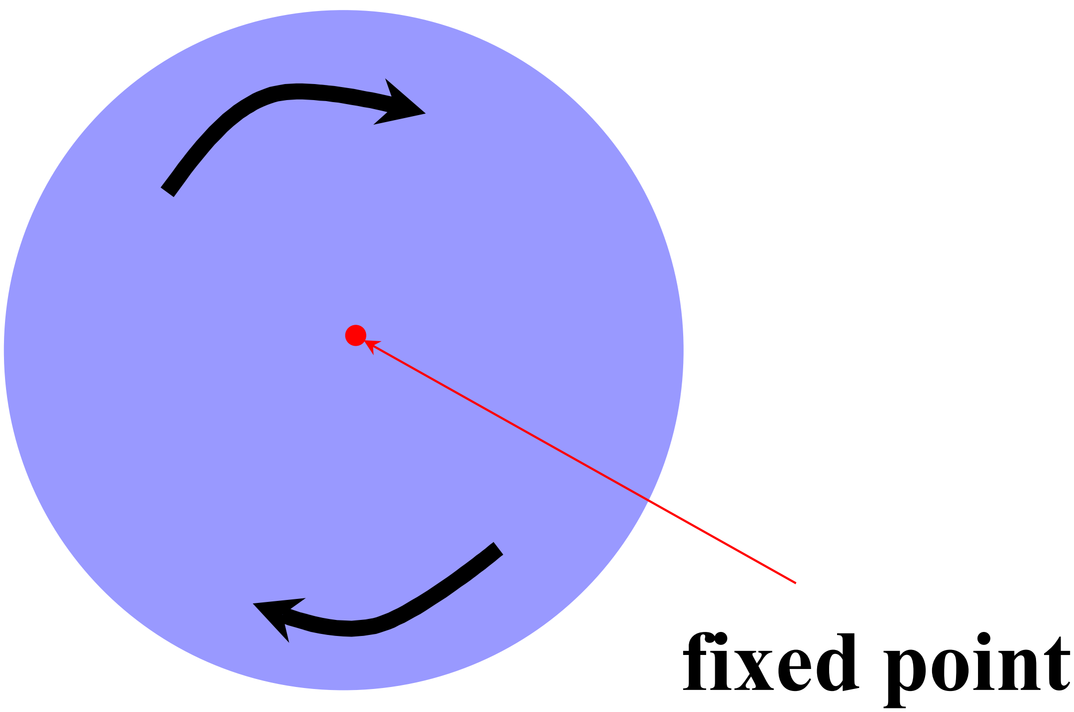
[Brouwer 1910]: Let $f : D \rightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

Then there exists an $x \in D$ s.t. $x = f(x)$.

A few examples, when *D* is the 2-dimensional disk.



Brouwer's Fixed Point Theorem



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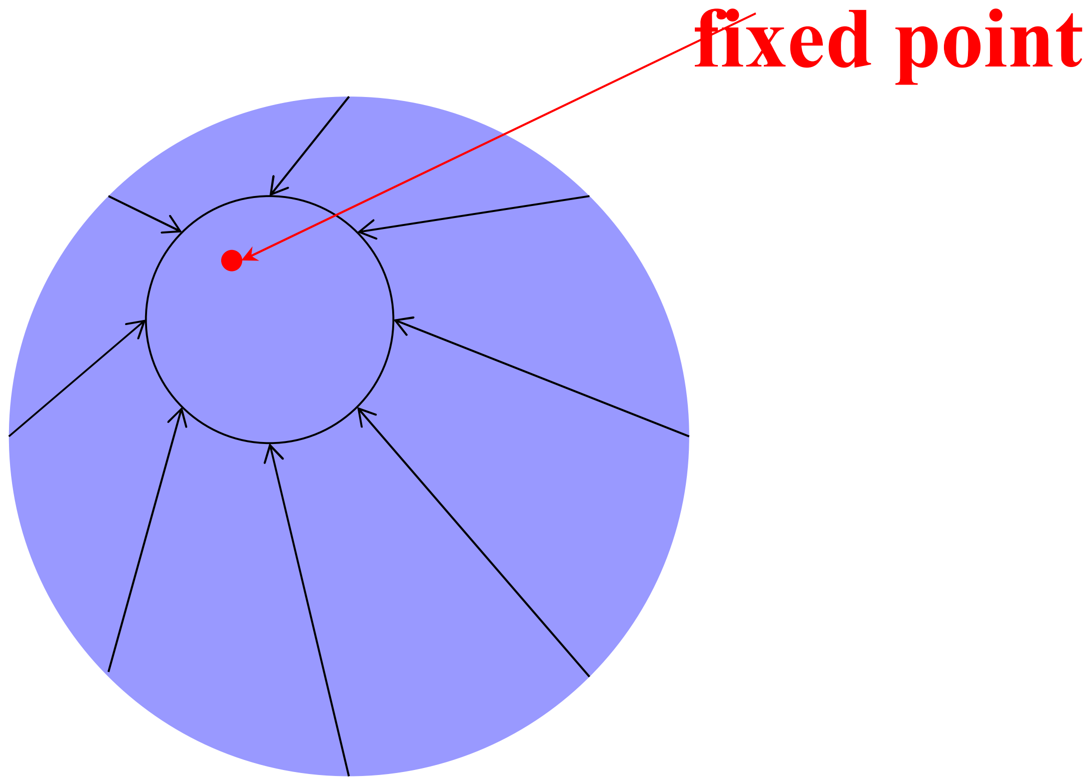

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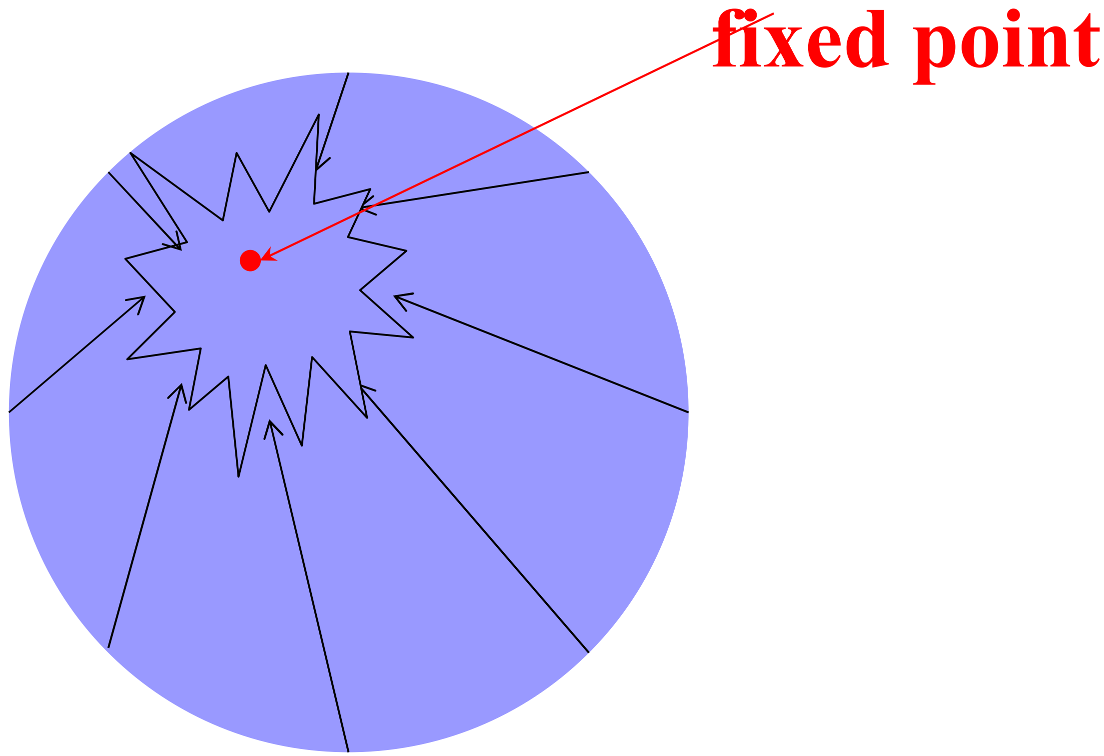

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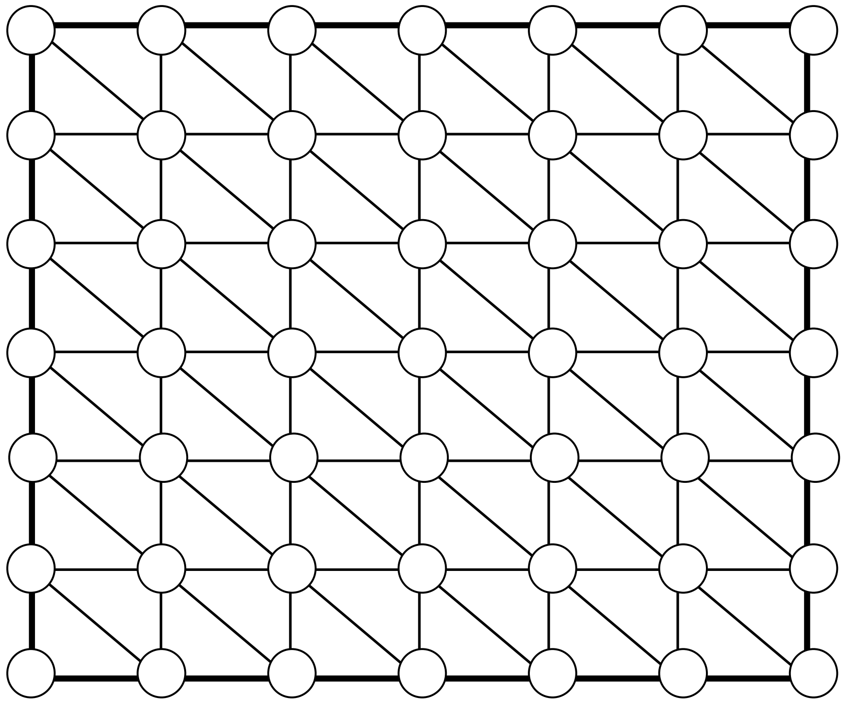
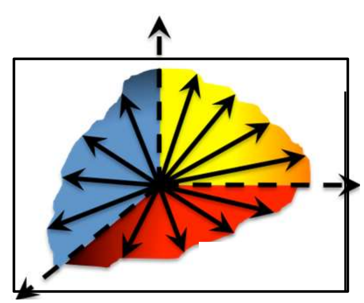
PPAD- Complete Existence Theorems: Brouwer, Sperner

Sperner's Lemma (2-d)





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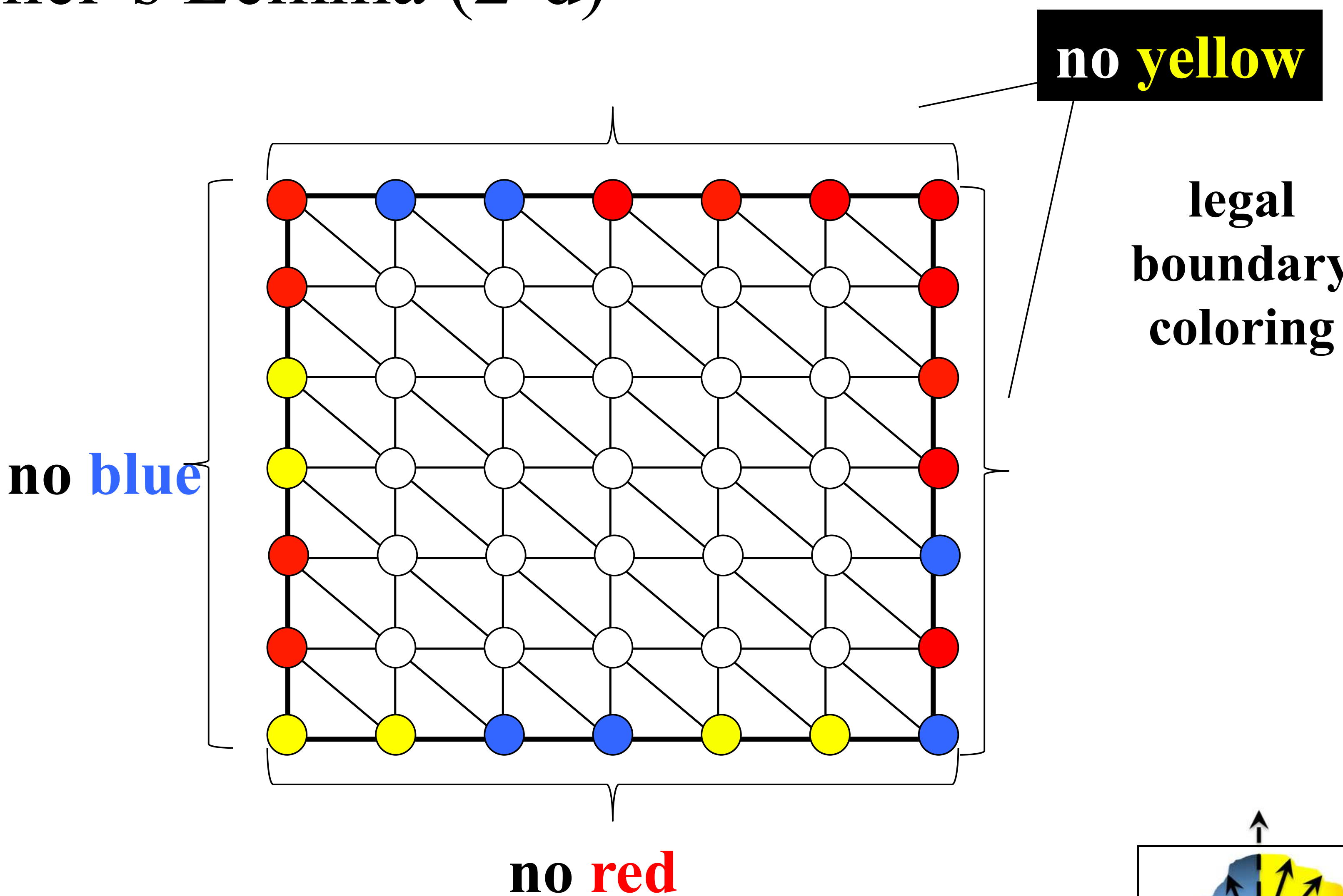
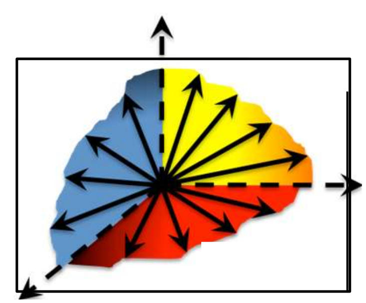


[Sperner 1928]: Color the boundary using three colors in a legal way.



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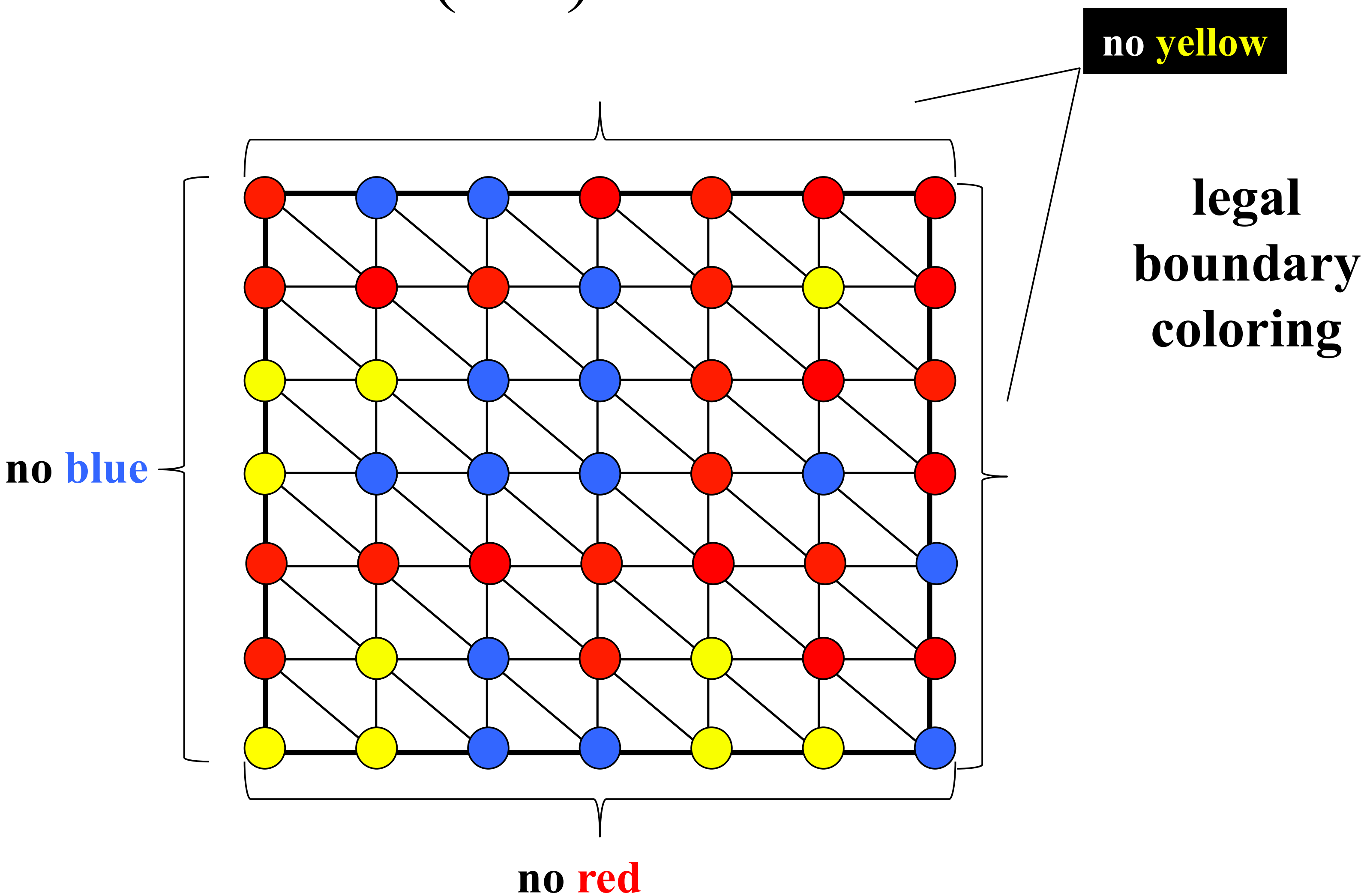
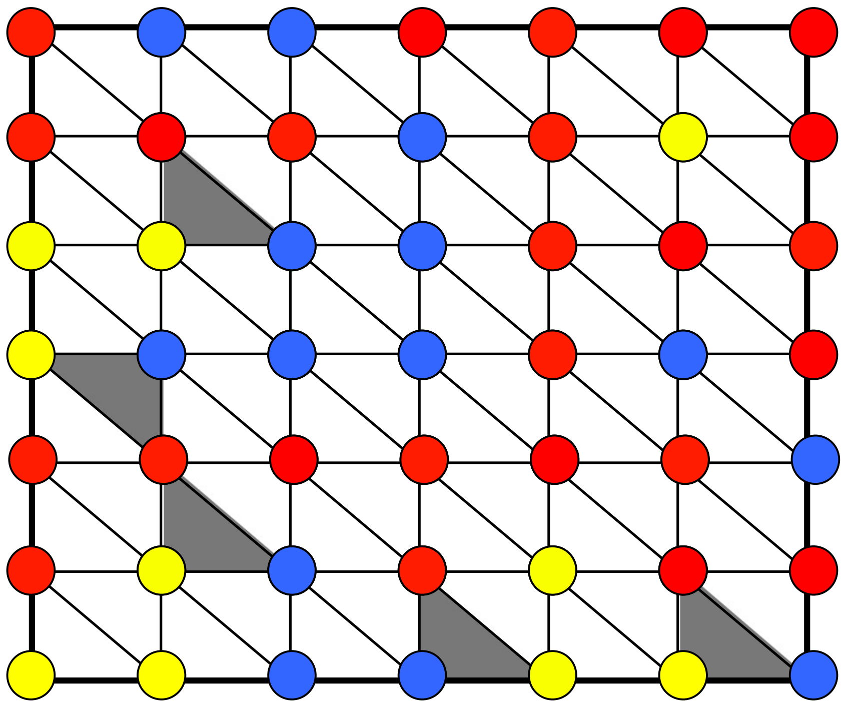


[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Sperner's Lemma (2-d)



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$Sperner \Rightarrow Brouwer$

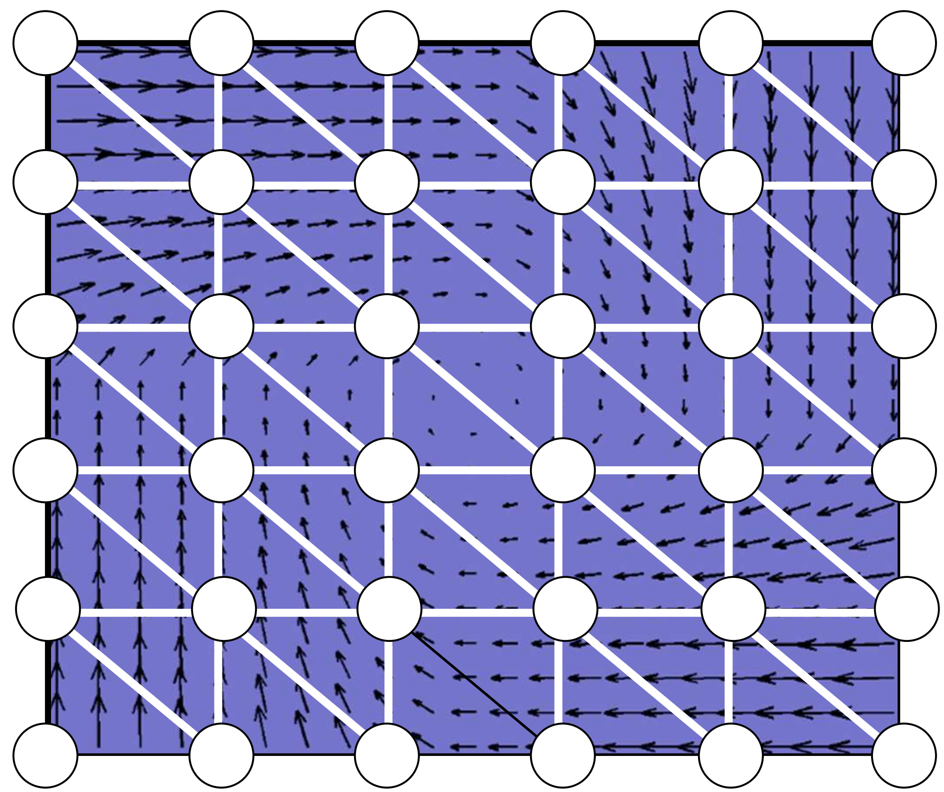
Sperner ⇒ Brouwer (High-Level)

Given $f : [0,1]^2 \to [0,1]^2$

1) For all $\epsilon > 0$, existence of approximate fixed point $|f(x)-x| < \epsilon$, can be shown via Sperner's lemma.

2) Then let $\epsilon \to 0$

For 1): Triangulate $[0,1]^2$;



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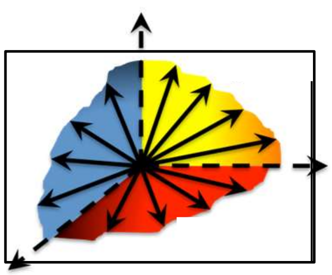

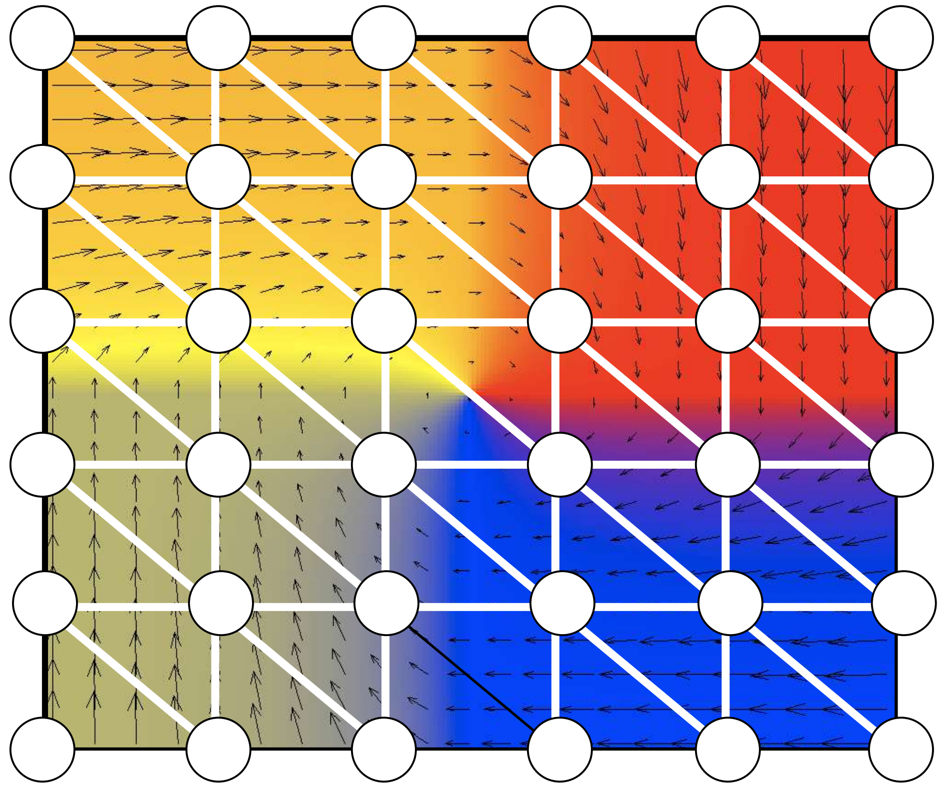

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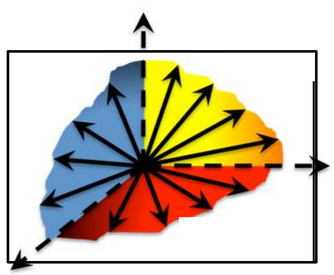

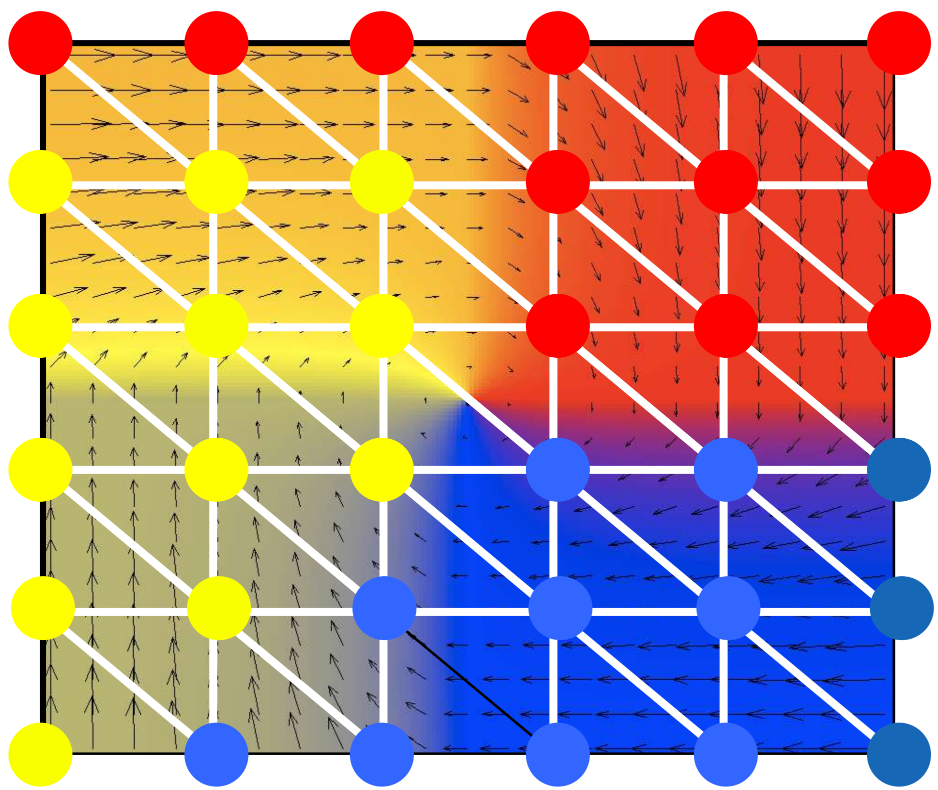

Sperner ⇒ Brouwer (High-Level)

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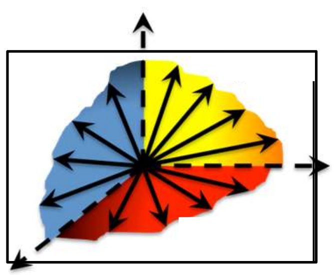
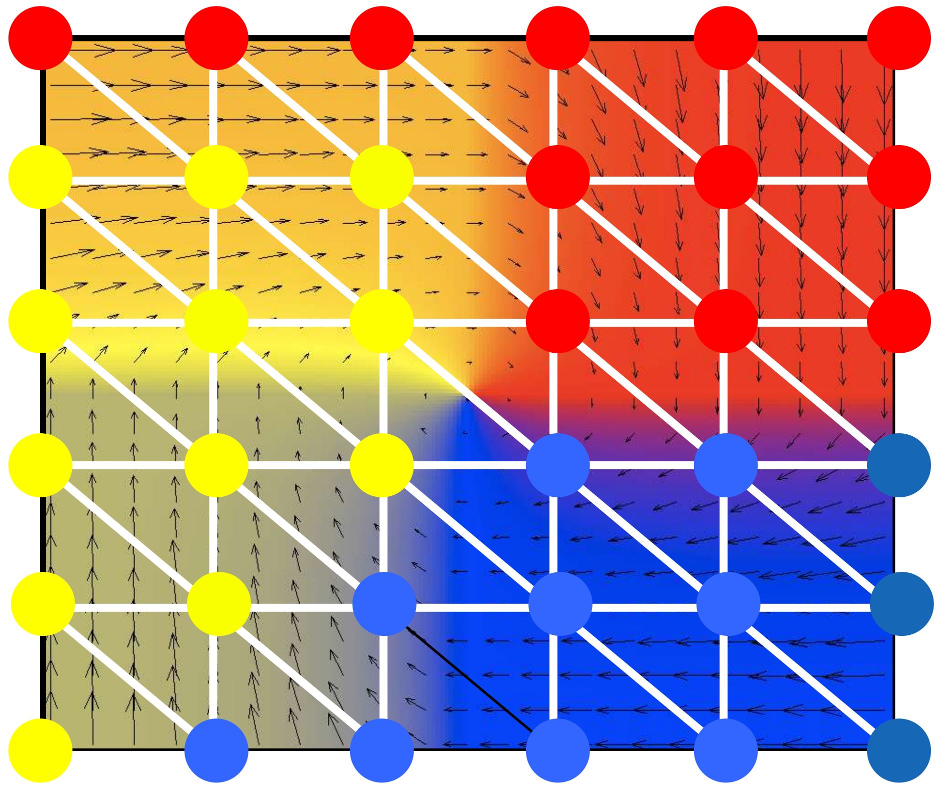
Apply Sperner.



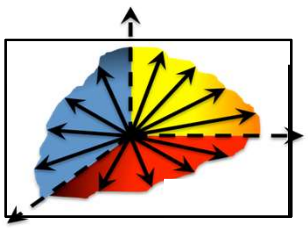


2D-Brouwer on the Square

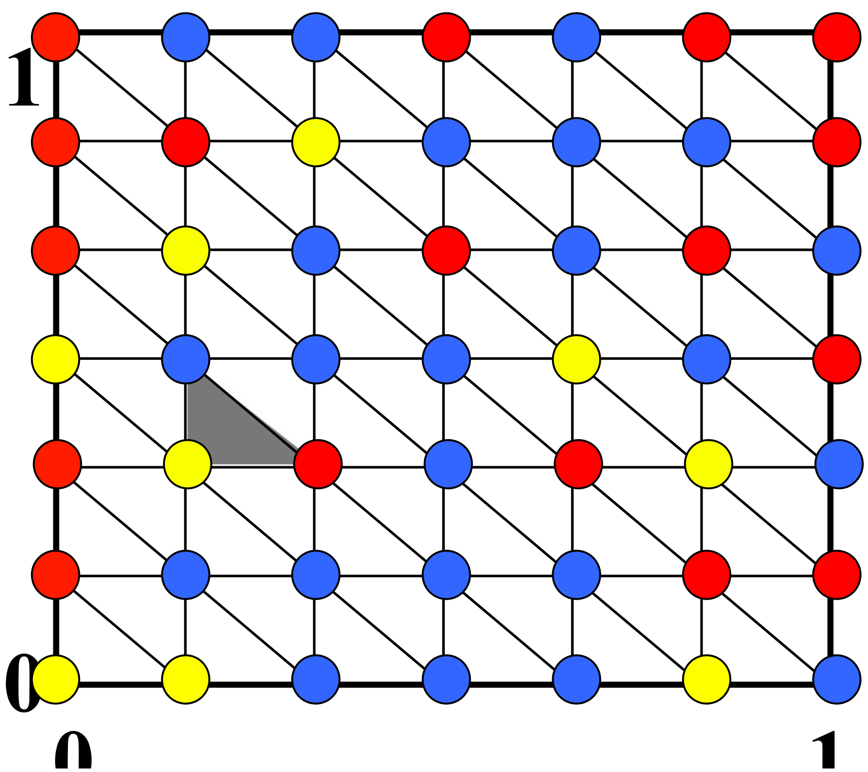
d **be** l_∞ **norm**

Suppose f**:** $[0,1]^2 \rightarrow [0,1]^2$**, continuous** **(by the Heine-Cantor theorem)**

$$\forall \epsilon, \exists \delta(\epsilon) > 0, s.t.$$
$$d(x, y) < \delta(\epsilon) \Rightarrow d\big(f(x), f(y)\big) < \epsilon$$



Choose small enough grid size so that..



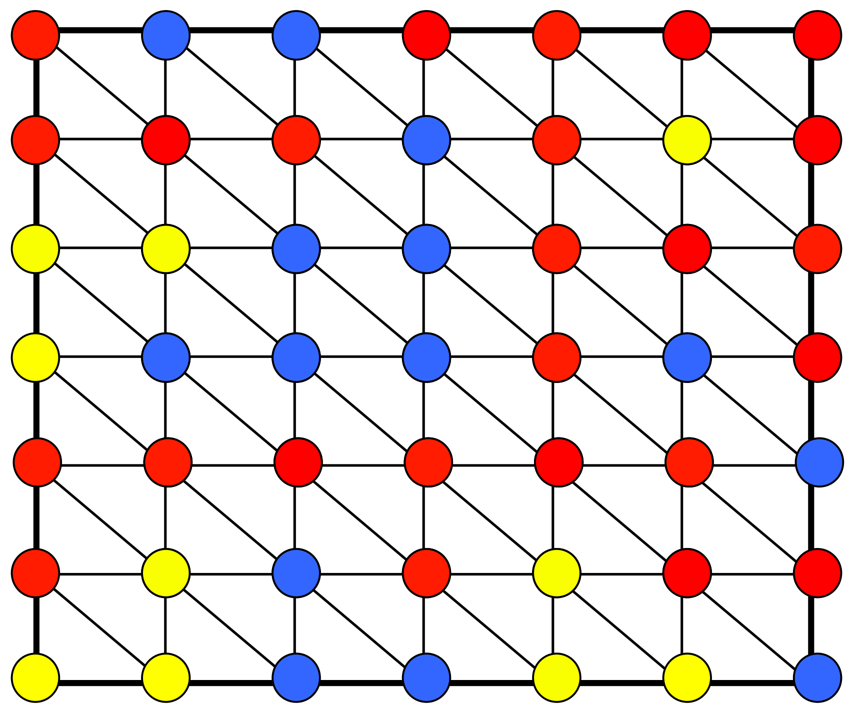
Claim: If z **a corner of a trichromatic triangle, then choosing** $\delta = \min\{\delta(\epsilon), \epsilon\}$

$$|f(z) - z|_\infty < c\delta, \qquad c > 0$$

Menu

- Existence Theorems: Brouwer, Sperner
- (Constructive) proof of Sperner → PPAD.

Proof of Sperner's Lemma



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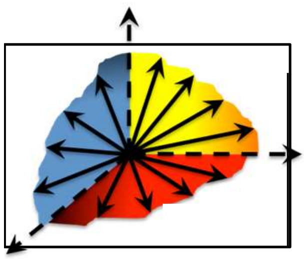

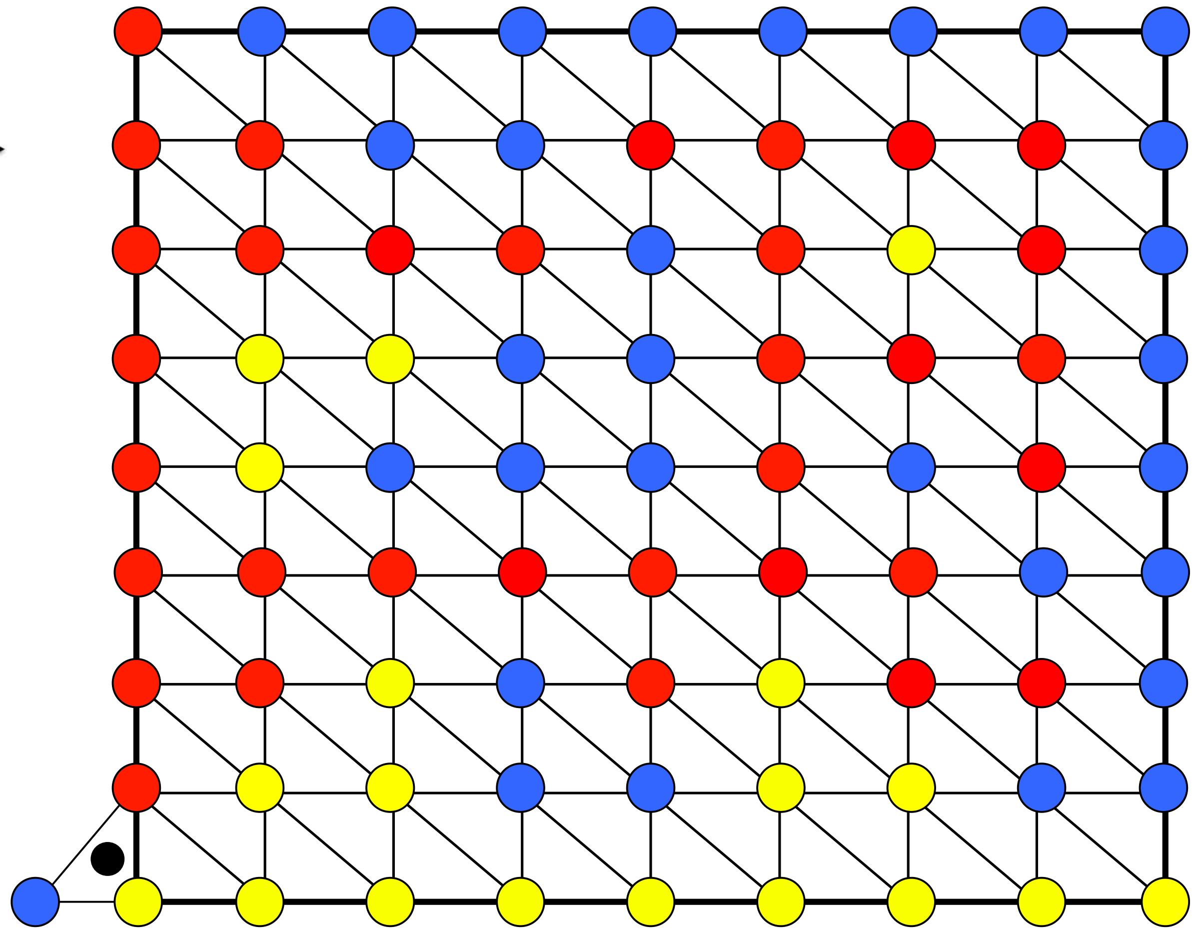

For convenience introduce an outer boundary, that does not create new tri-chromatic triangles.

Also introduce an artificial tri-chromatic triangle.

Next define a directed walk starting from the artificial tri-chromatic triangle.

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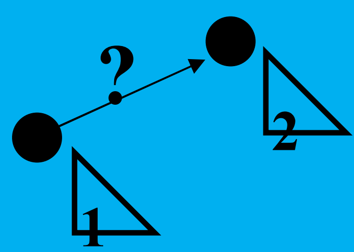
Proof of Sperner's Lemma

Space of Triangles

***Transition Rule*:**

If ∃ red - yellow door cross it with red on your left hand.



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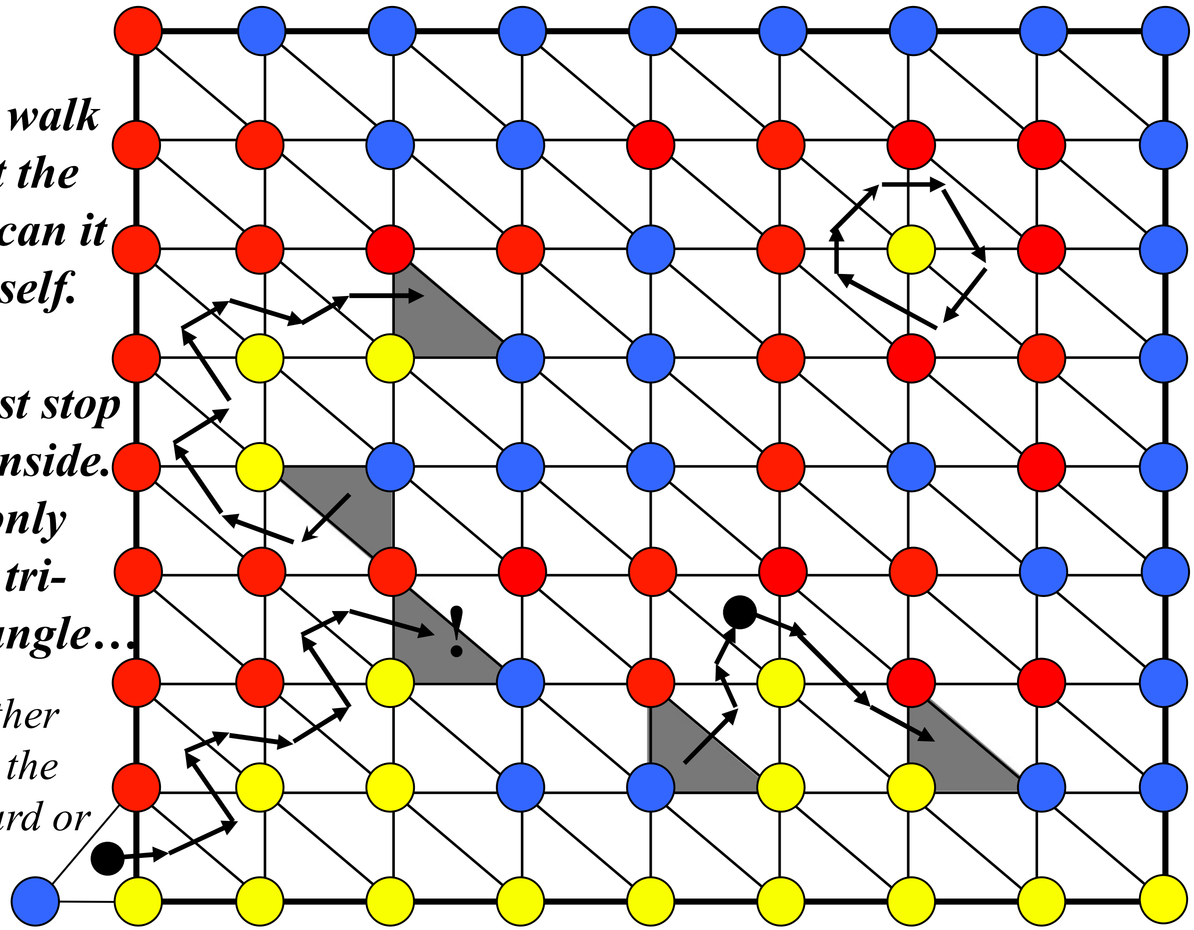
Proof of Sperner's Lemma

Claim: ***The walk cannot exit the square, nor can it loop into itself.***

Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle…

Starting from other triangles we do the same going forward or backward.



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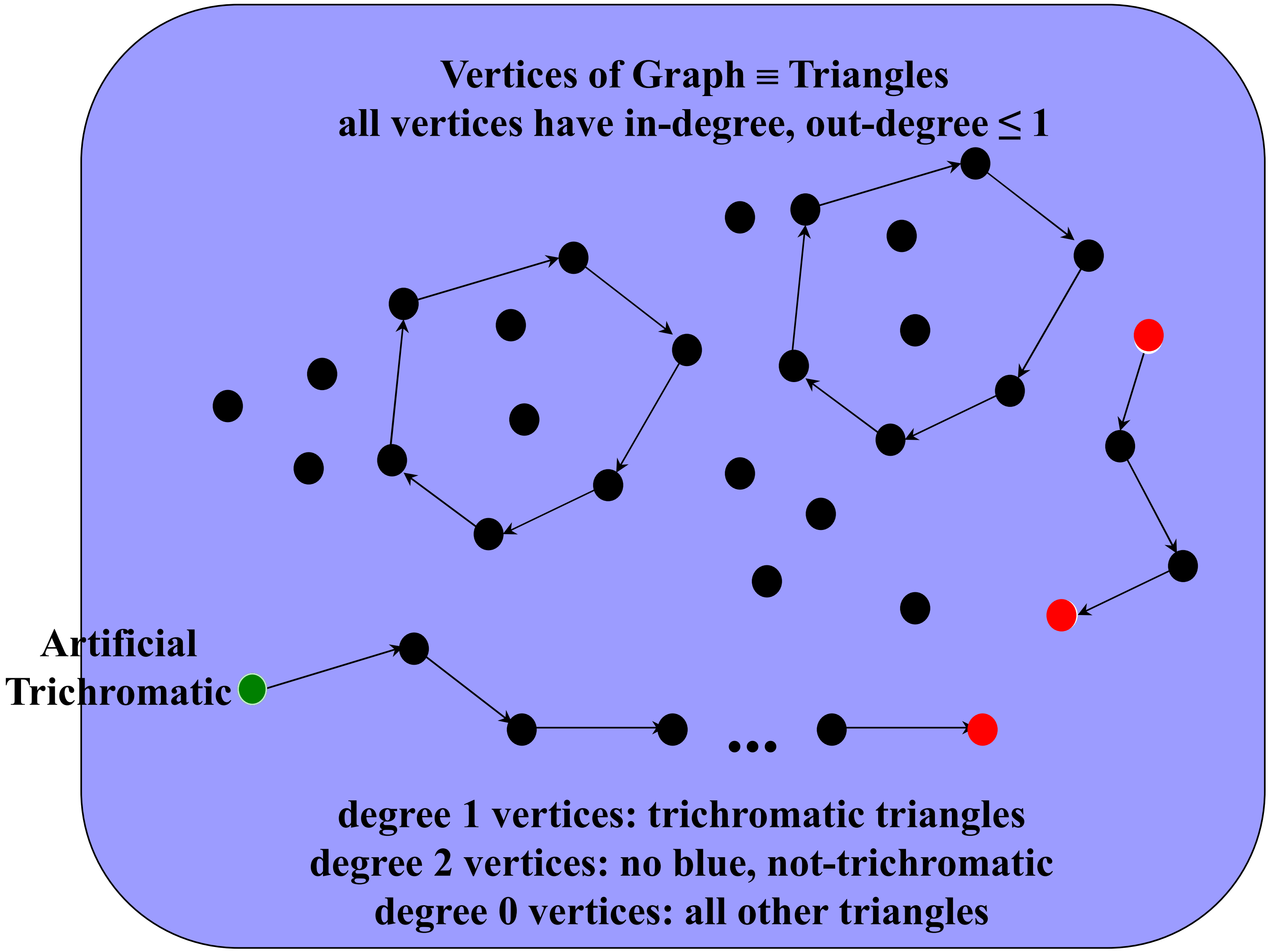
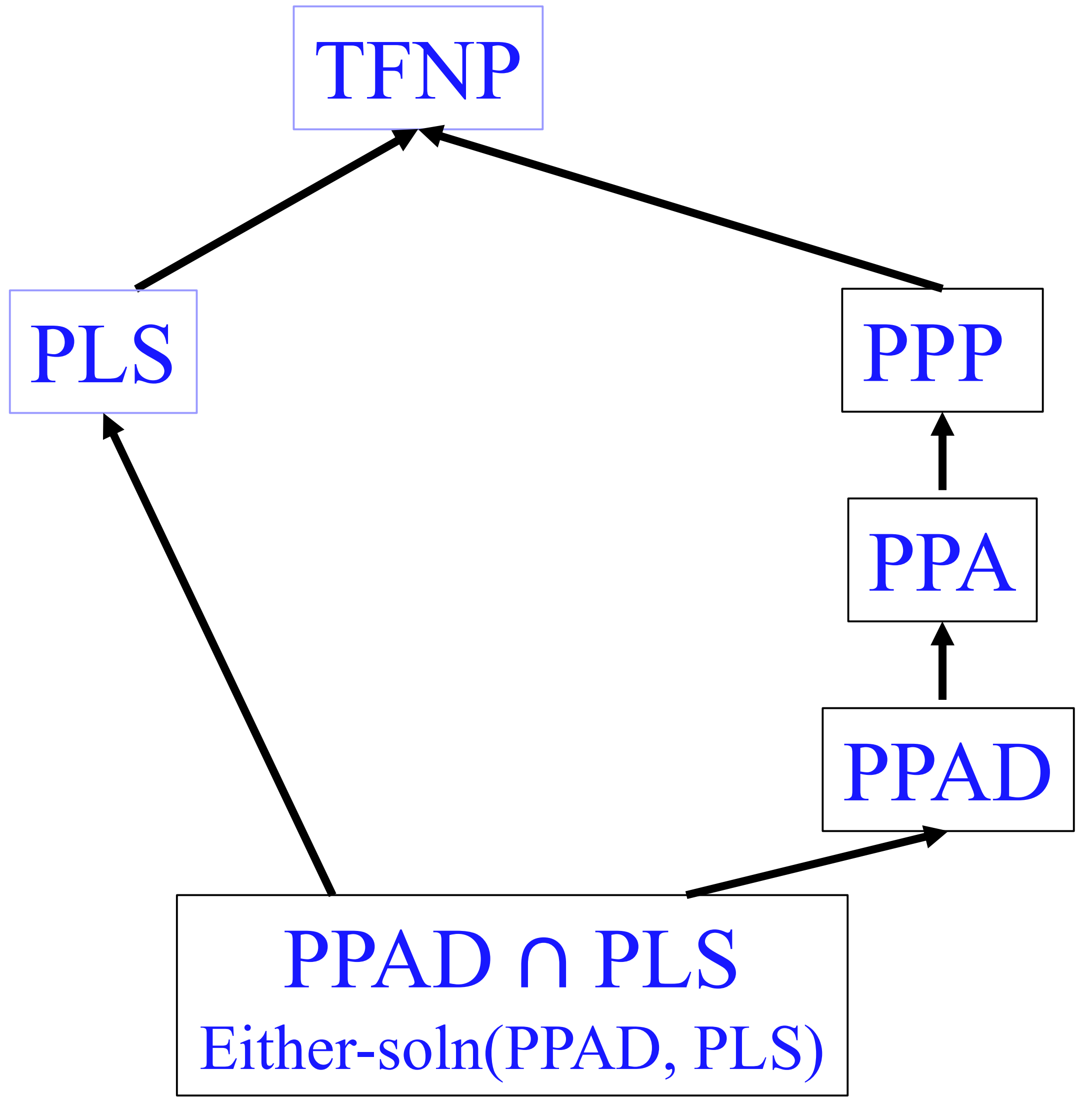
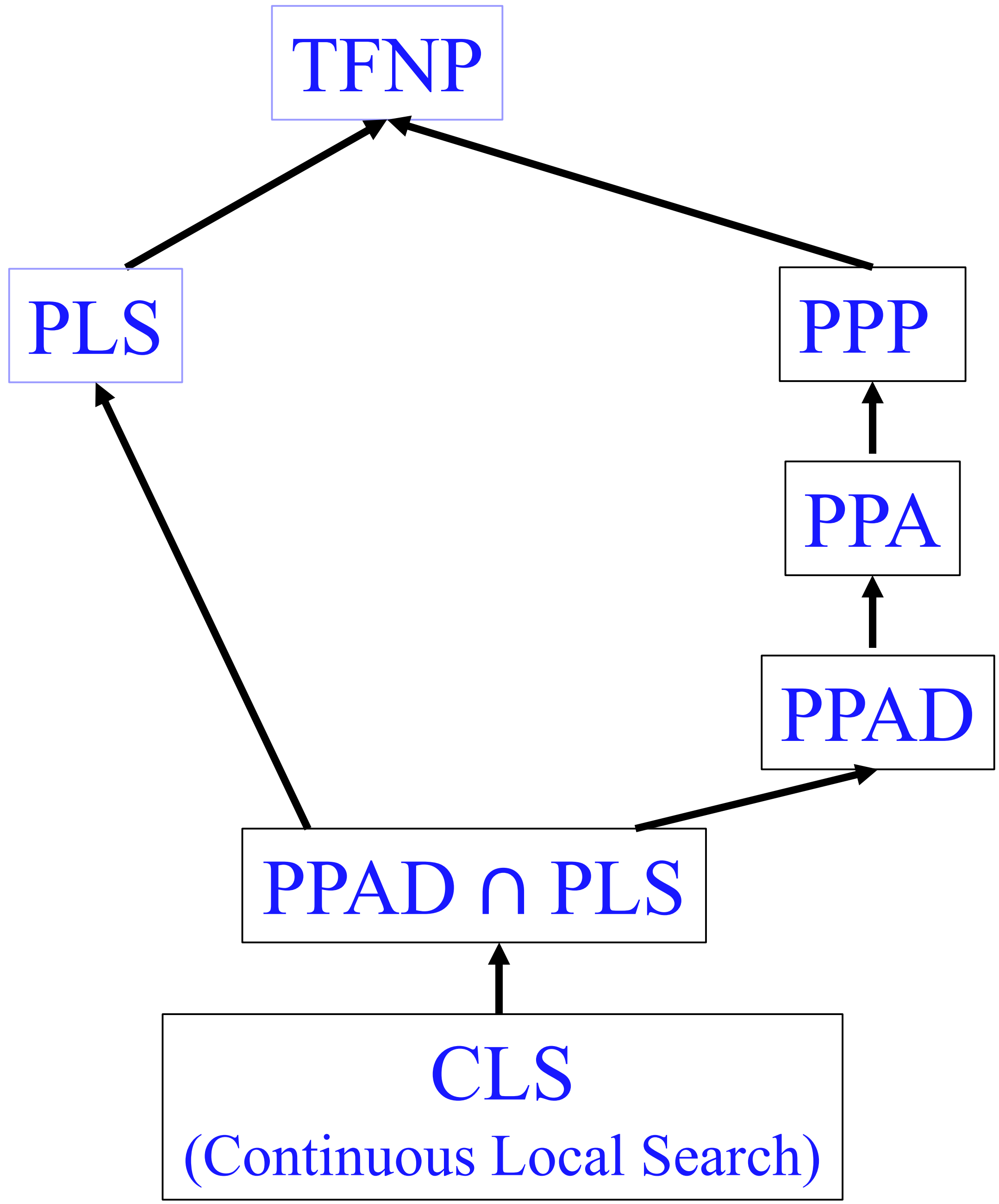
Proof Structure: A directed parity argument



Proof: ∃ at least one trichromatic (artificial one) ➔ ∃ another trichromatic


TFNP
PLS
PPP
PPA
PPAD
PPAD ∩ PLS
Either-soln(PPAD, PLS)



TFNP
PLS
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PPAD ∩ PLS
CLS
(Continuous Local Search)

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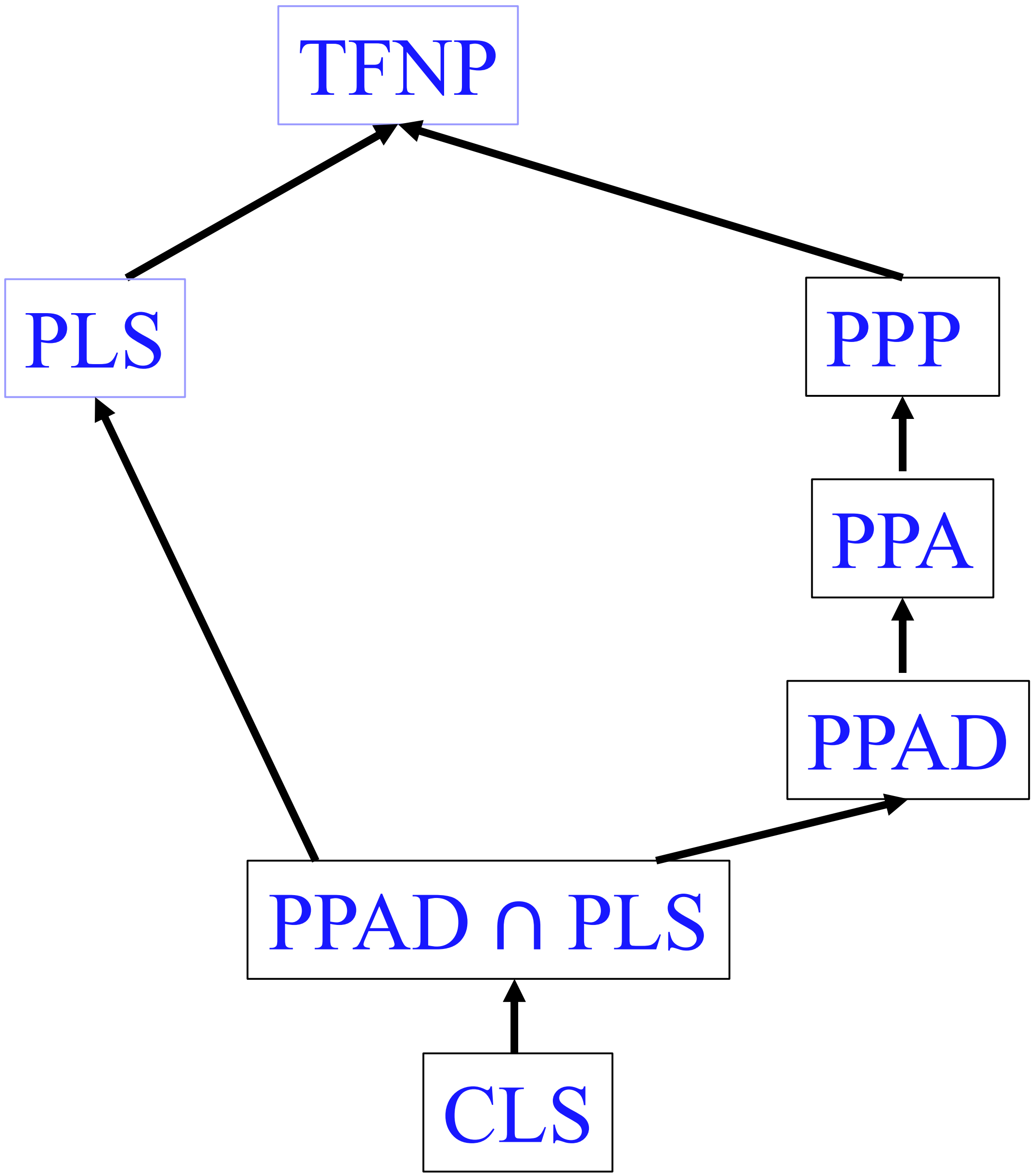
TFNP
PLS
PPP
PPA
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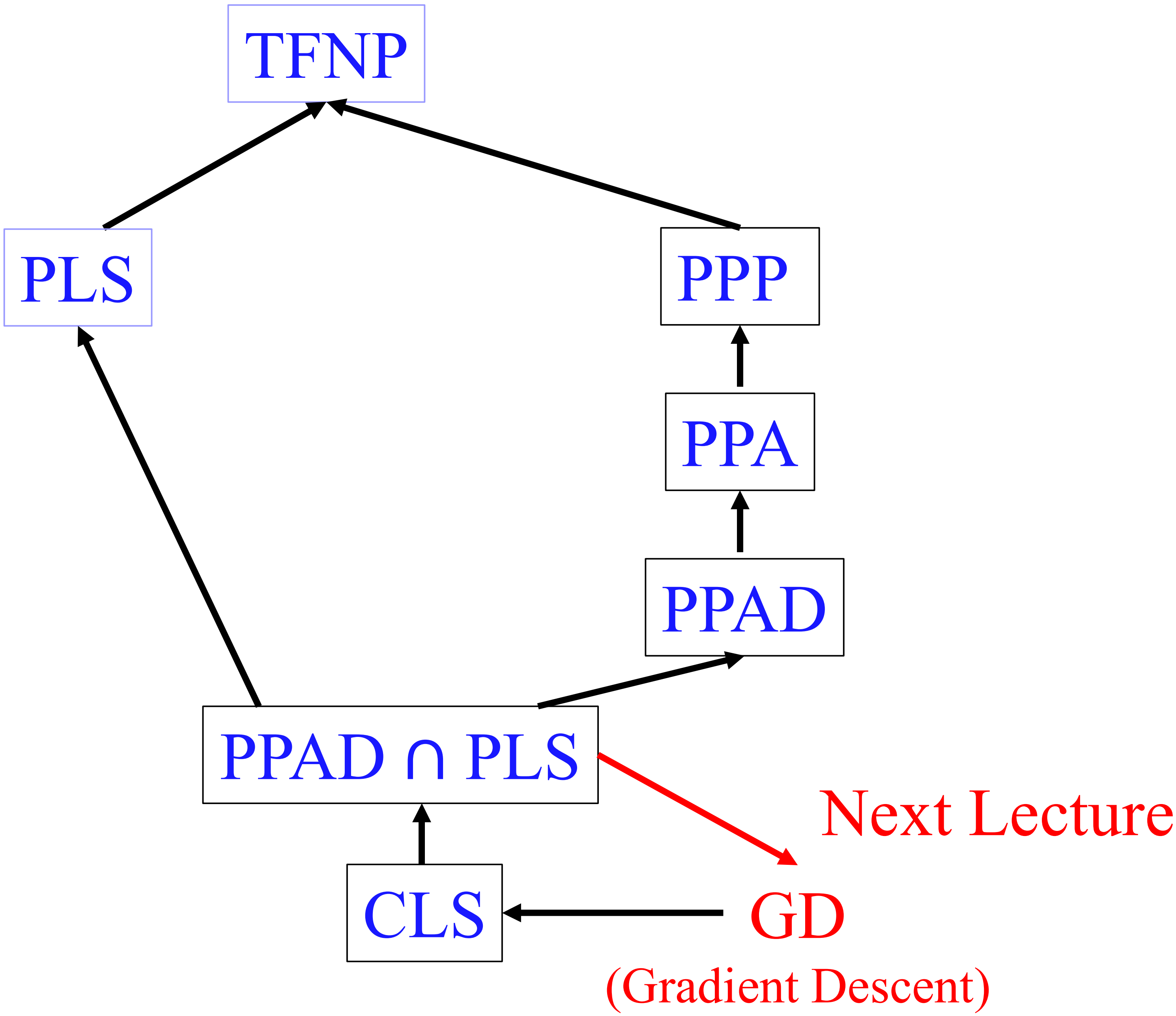
CLS



TFNP
PLS
PPP
PPA
PPAD
PPAD ∩ PLS
Next Lecture
CLS
GD
(Gradient Descent)