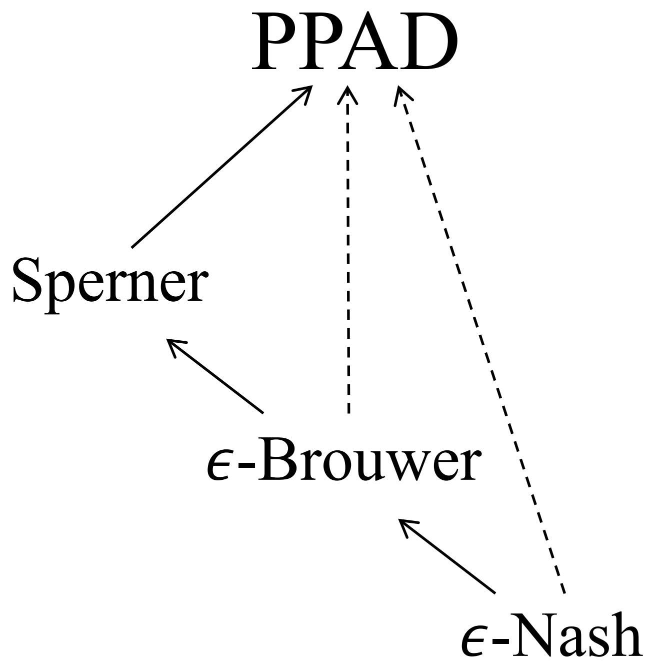
PPAD-hardness for Two-Player Games

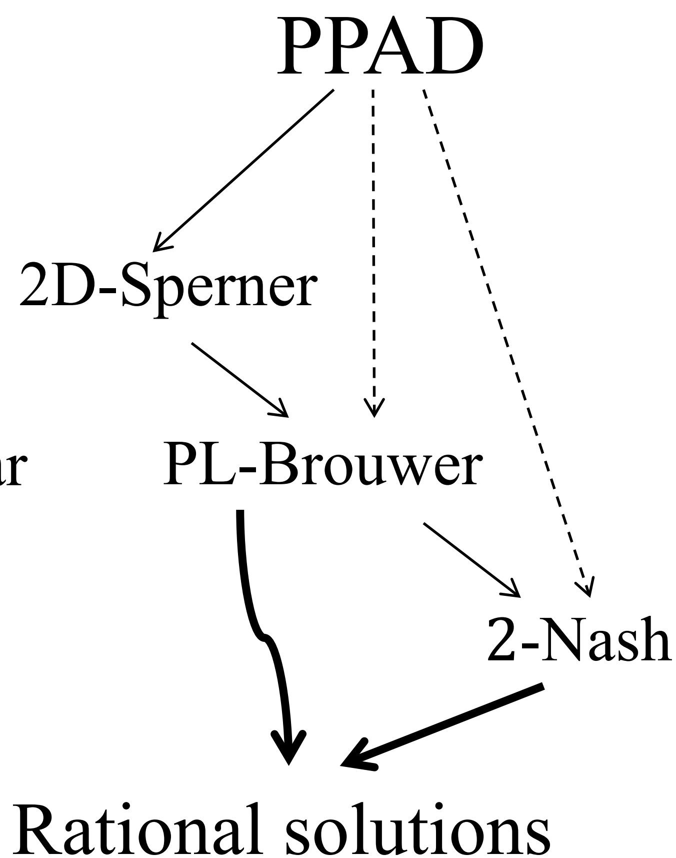
CS 598RM

Ruta Mehta

Previous Lecture

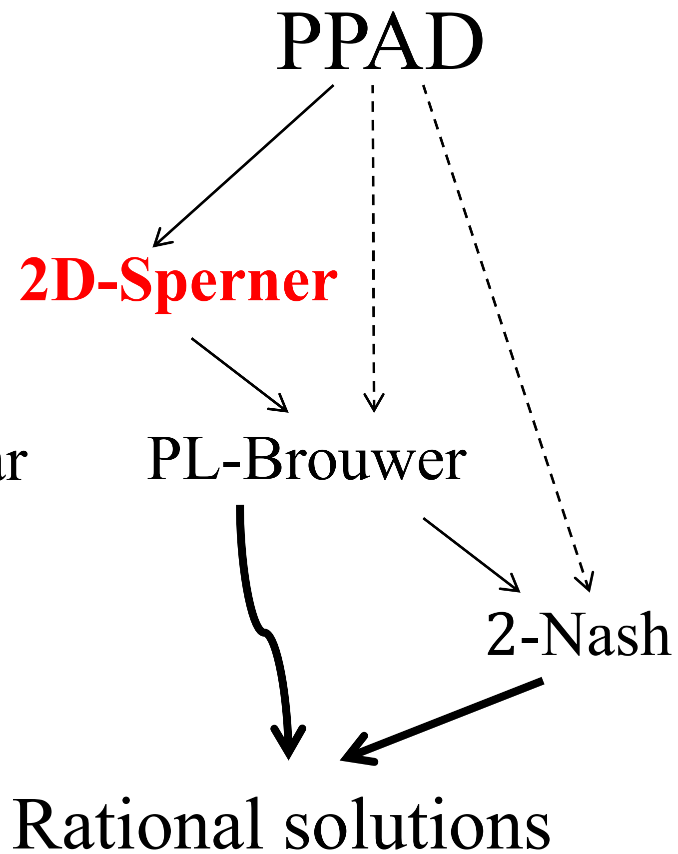


Today



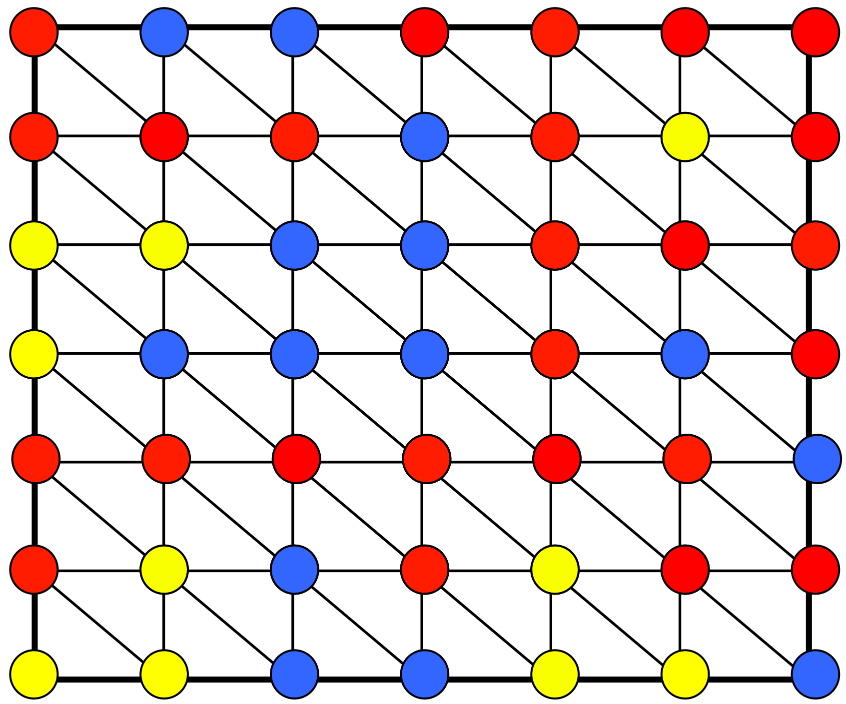
PL: Piecewise-Linear

Today



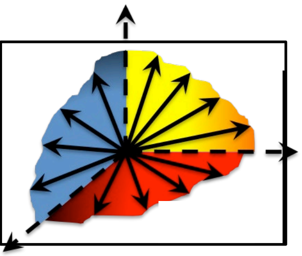
PL: Piecewise-Linear

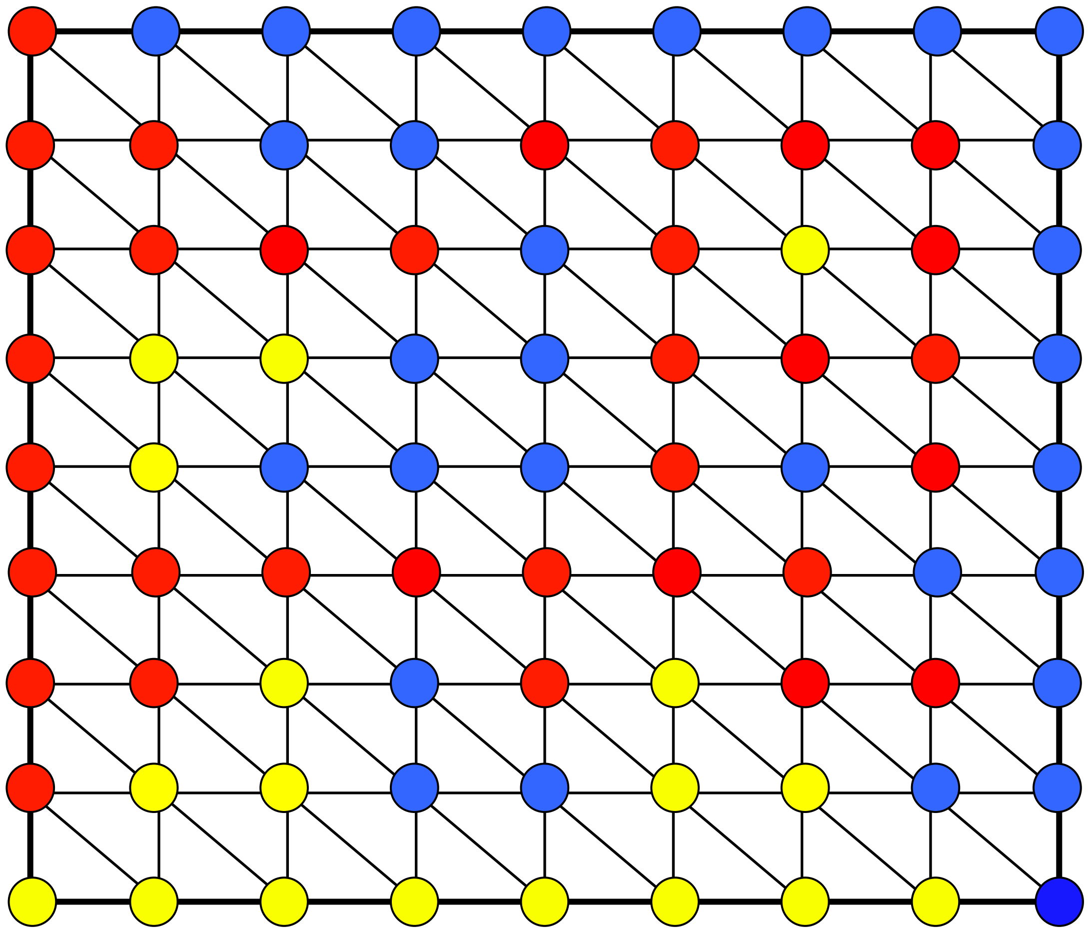
Recall: 2D-Sperner



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Recall: Proof of 2D-Sperner's Lemma

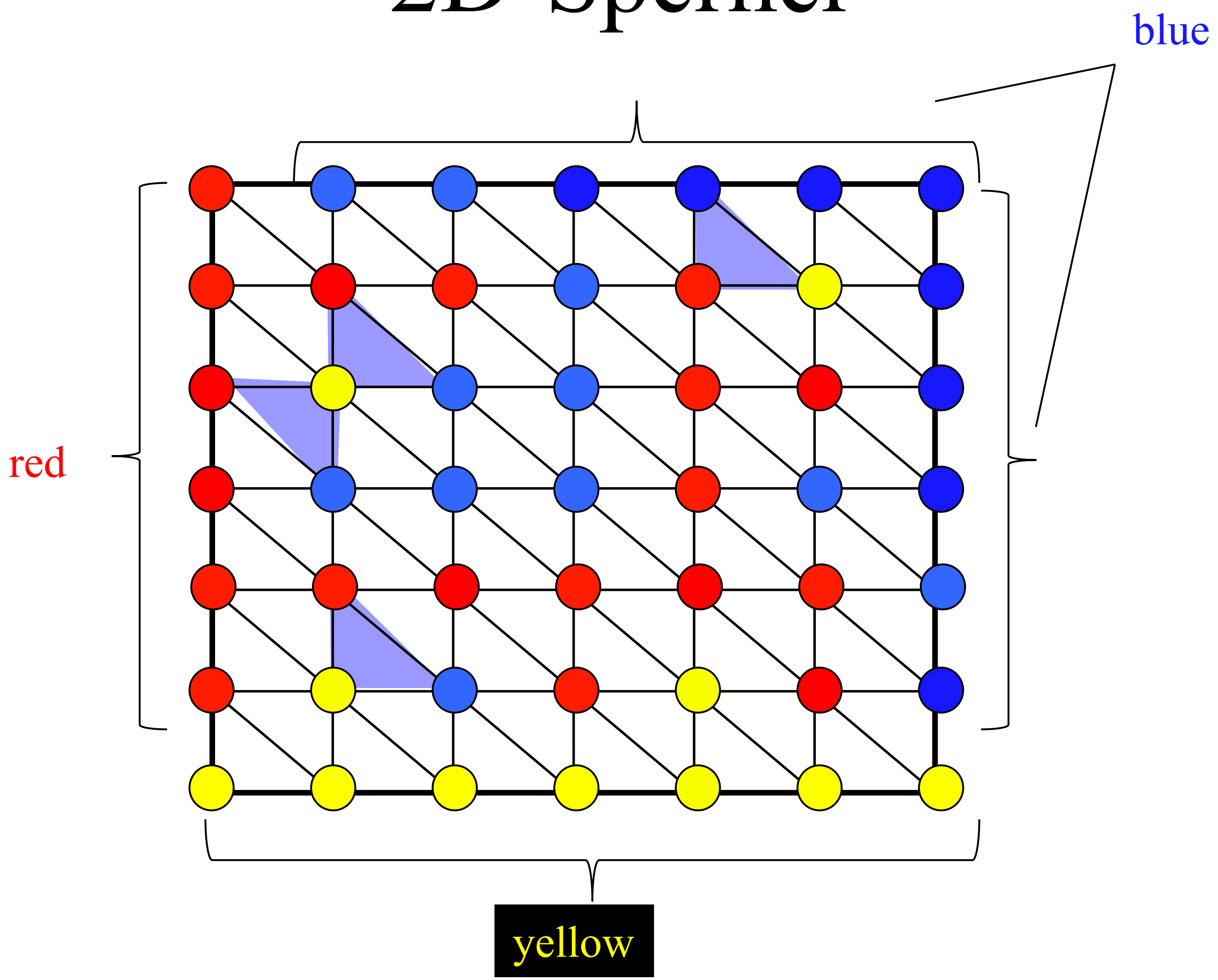




For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

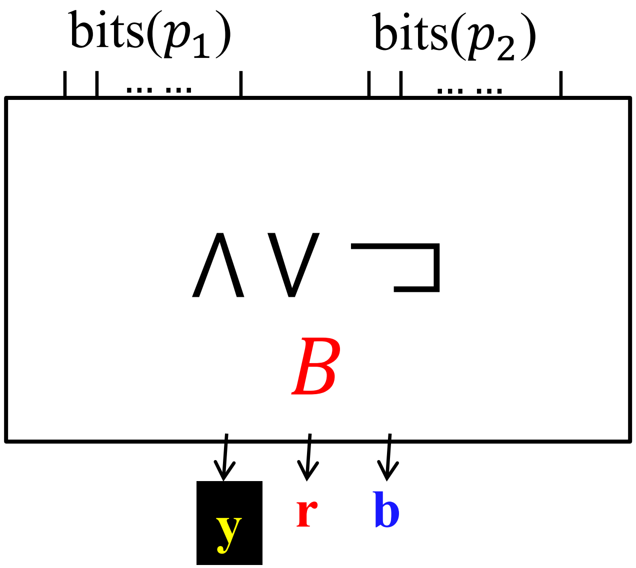
[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

2D-Sperner

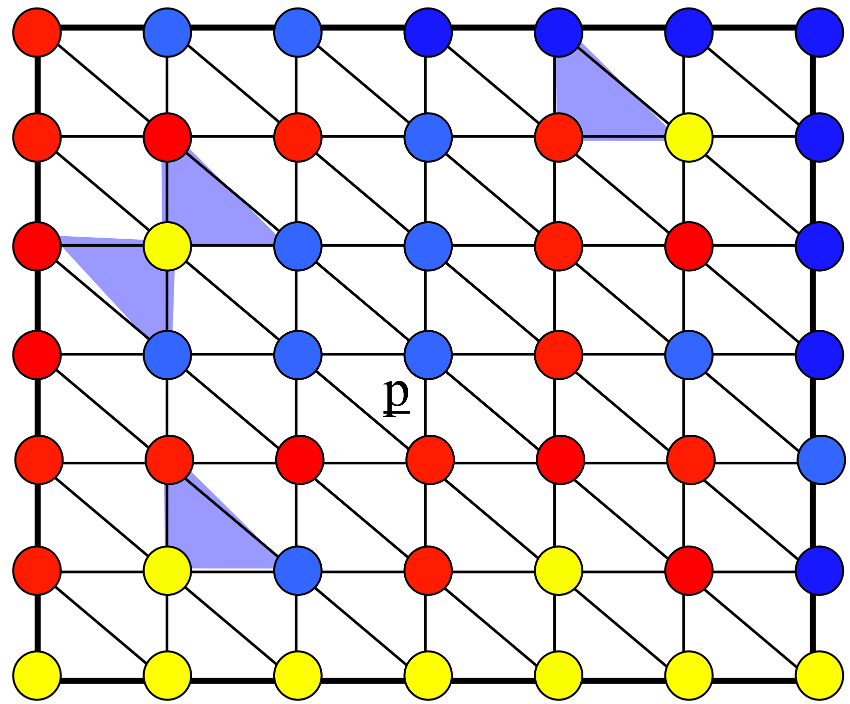


Lemma: Color the boundary in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle.

2D-Sperner

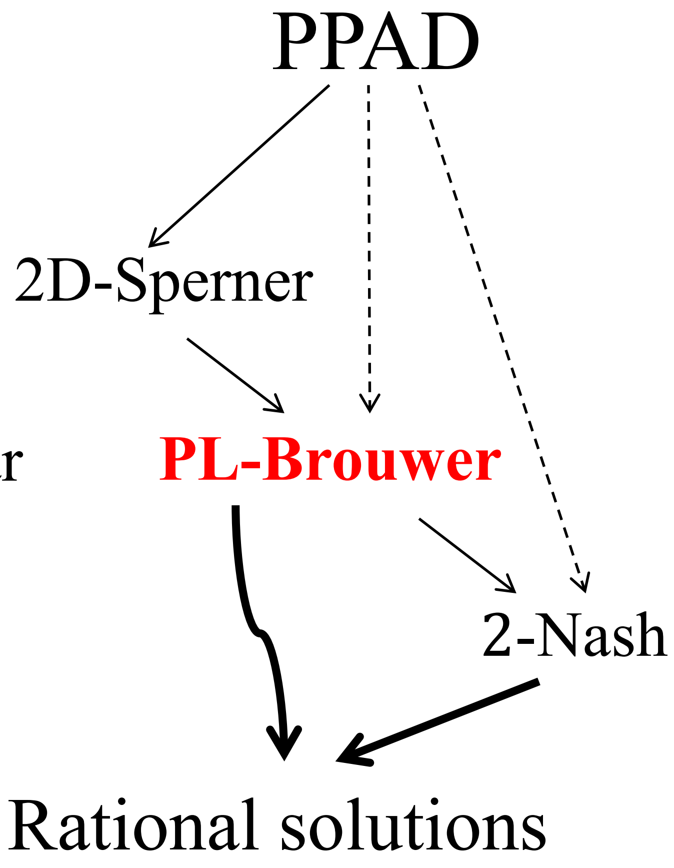


Only one is non-zero



Lemma: Color the boundary in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle.

Today

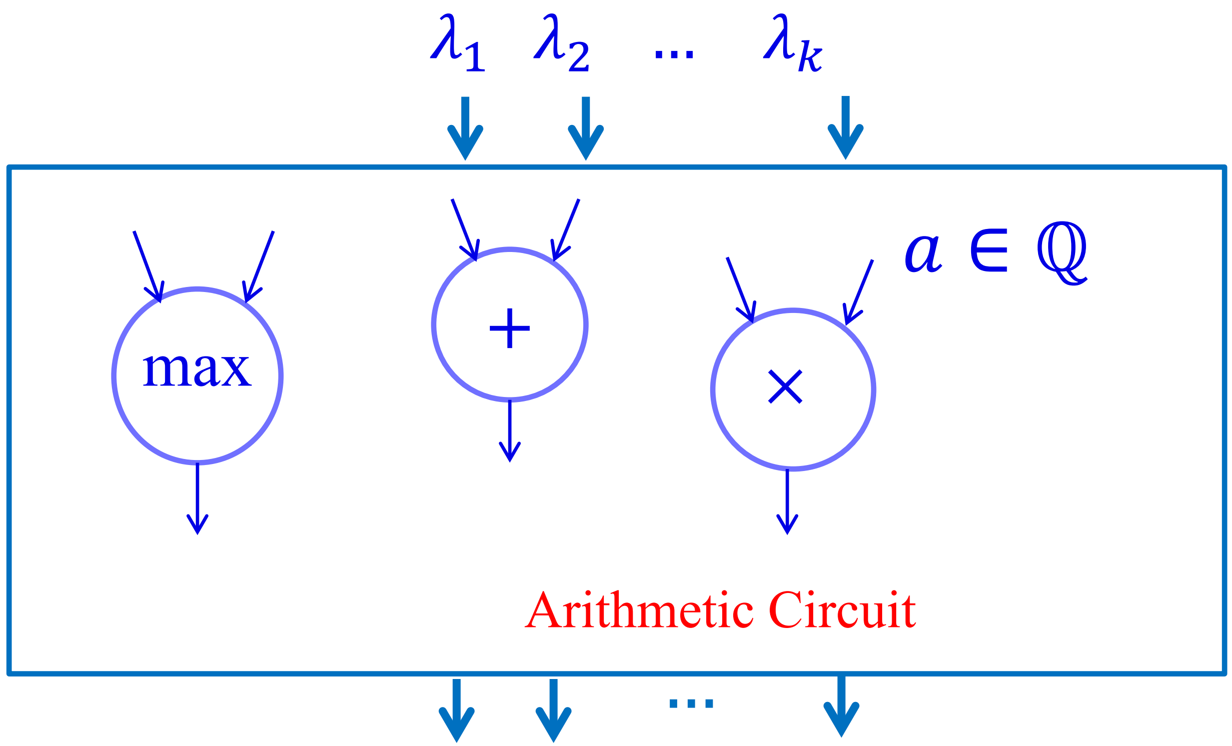


PL: Piecewise-Linear

PL-Brouwer (Linear-FIXP)

$F: [0,1]^k \to [0,1]^k$

Find a fixed Point of F:
$x \in [0,1]^k$ s.t.
$F(x) = x$



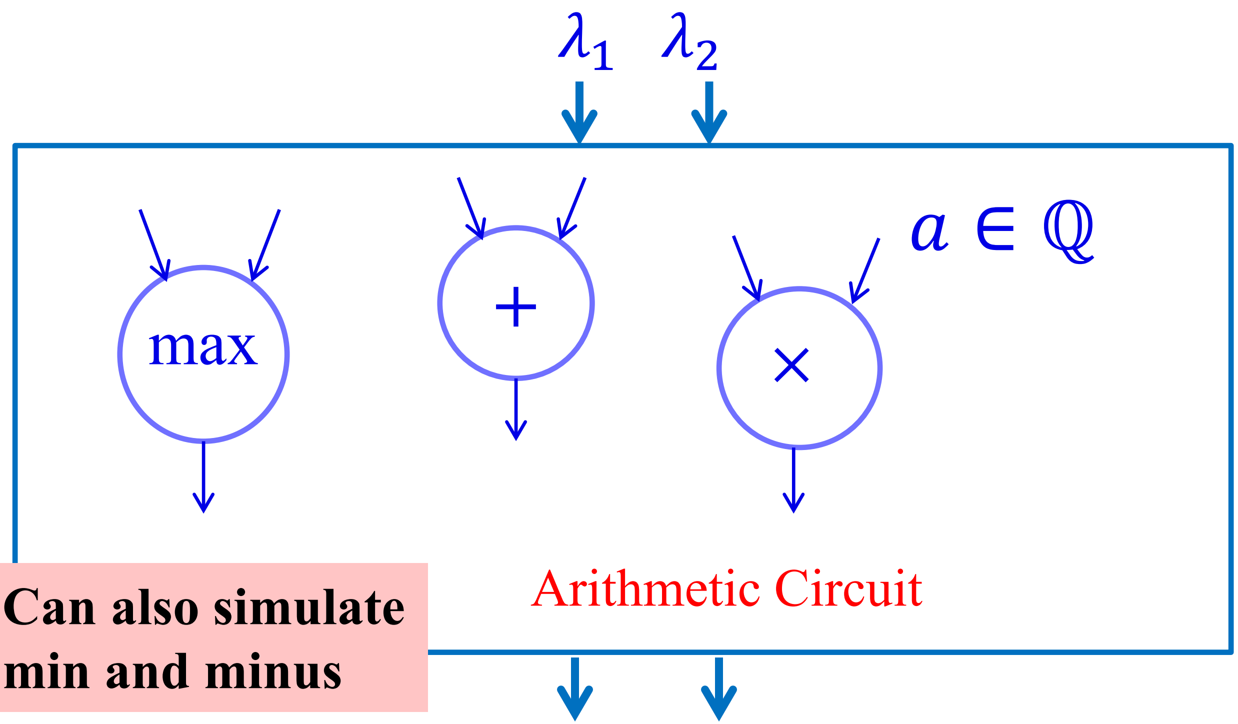
Continuous piecewise-linear function

PL-Brouwer (Linear-FIXP)

(EY'07) $F: [0,1]^2 \rightarrow [0,1]^2$

We need only 2-D

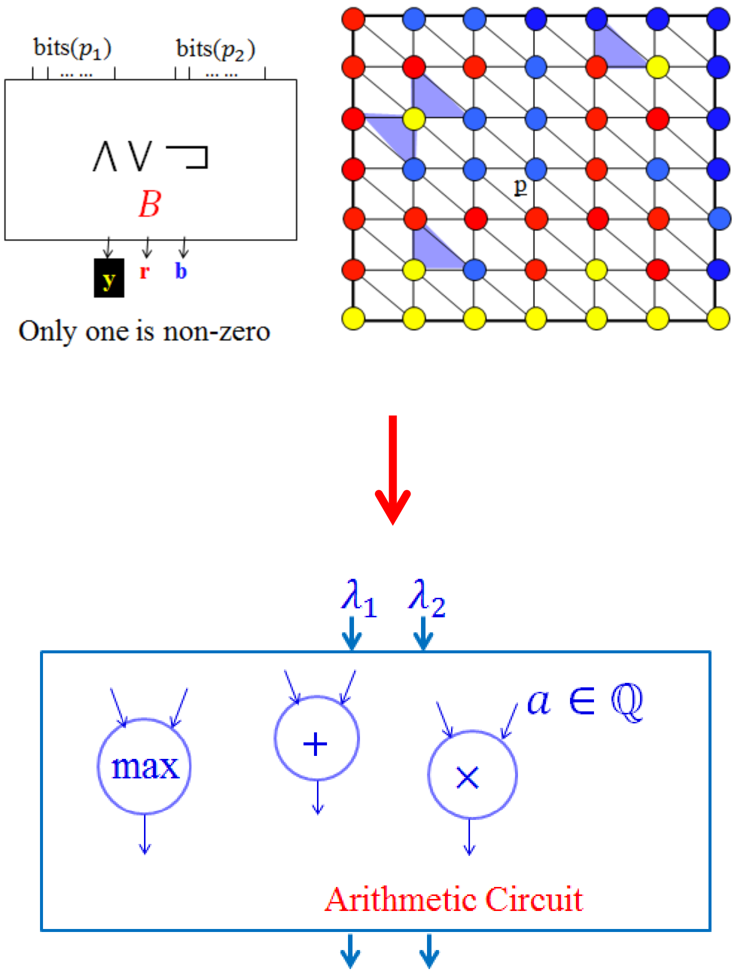
Find a fixed Point of F: $x \in [0,1]^k$ s.t. $F(x) = x$

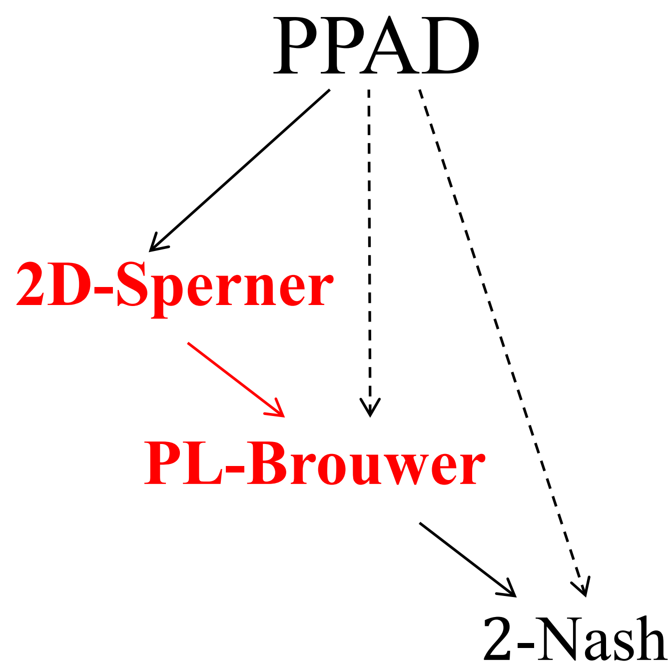


Can also simulate min and minus

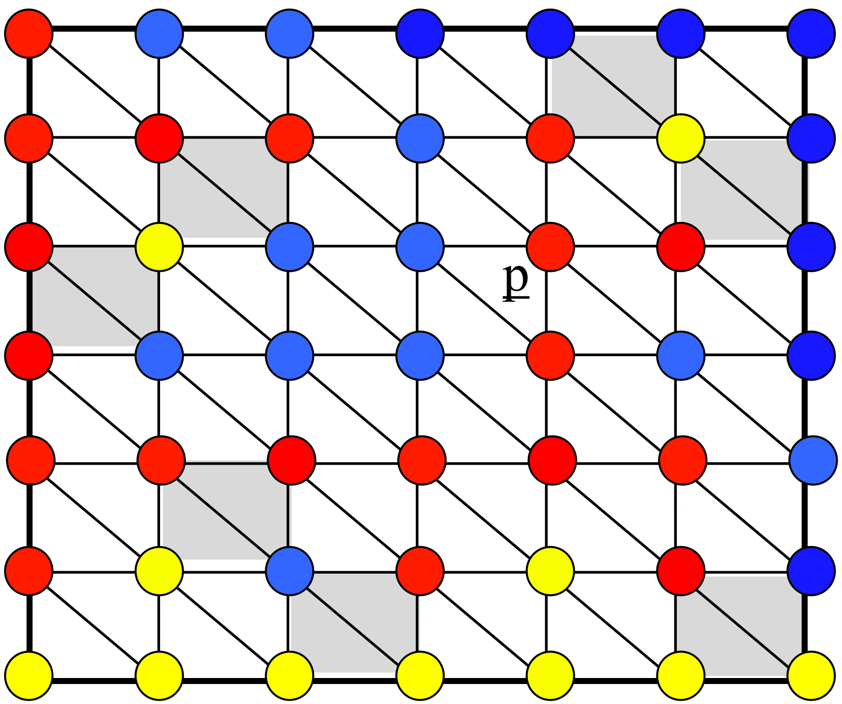
Continuous piecewise-linear function

Today


bits(p_1)
bits(p_2)
$\wedge \vee \neg$
B
y r b
Only one is non-zero
p
λ_1 λ_2
max
+
×
$a \in \mathbb{Q}$
Arithmetic Circuit


PPAD
2D-Sperner
PL-Brouwer
2-Nash

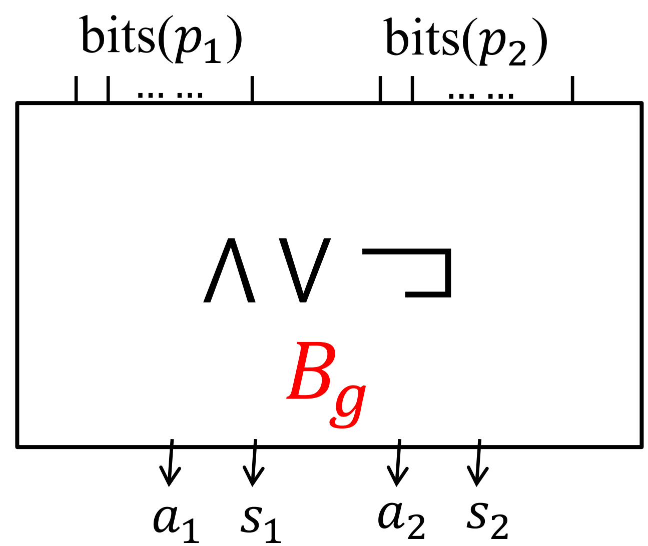
2D-Sperner: As a Discrete Function



g(p) = p + D(p)

D(●) = (-1, -1)
D(●) = (1, 0)
D(●) = (0, 1)

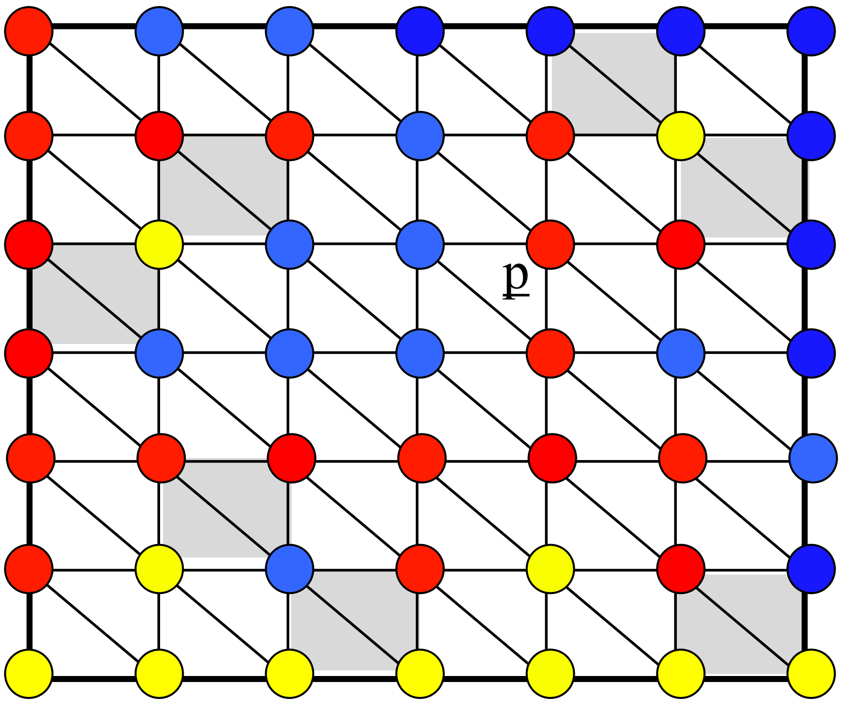
Boolean Circuit



D(p) = $(a_1 - s_1,\ a_2 - s_2)$

Given B_g find a tri-chromatic square.

2D-Sperner: As a Discrete Function

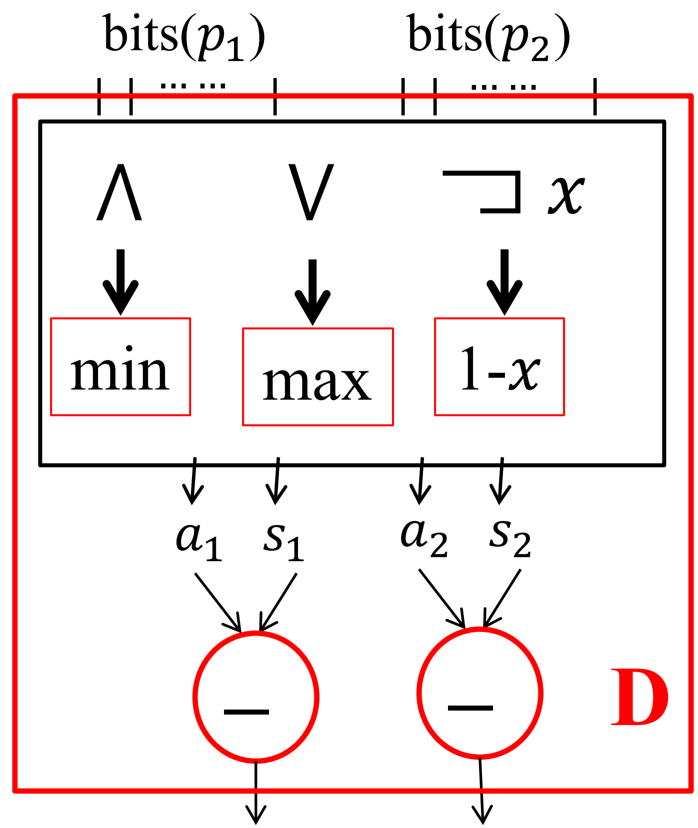


$\mathbf{g}(\underline{p}) = \underline{p} + D(\underline{p})$

D(●) = (-1, -1)

D(●) = (1, 0)

D(●) = (0, 1)

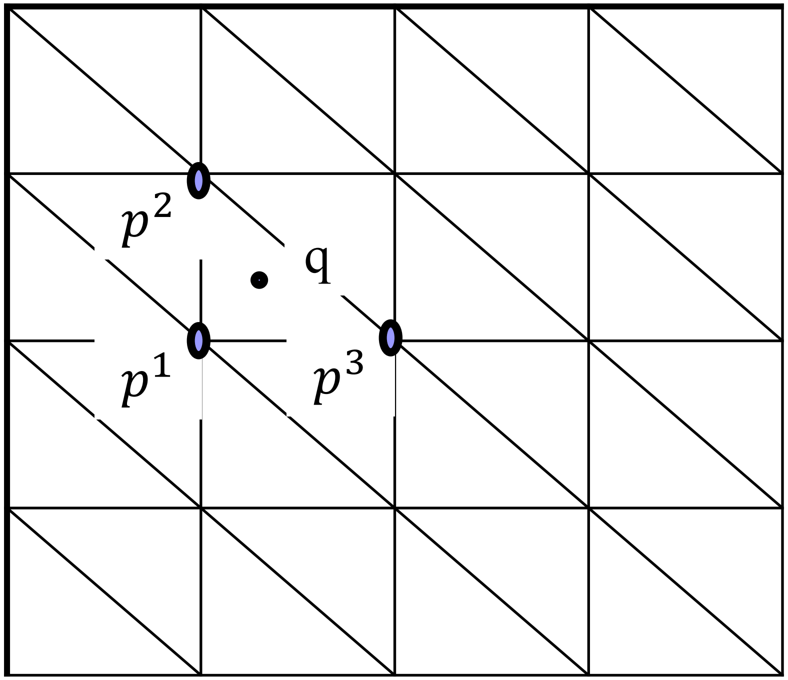


$D(\underline{p}) = (a_1 - s_1,\ a_2 - s_2)$

Arithmetic

Circuit for D

g → Continuous Func. (A possibility)



$\mathbf{g}(\underline{p}) = \underline{p} + I(\underline{p})$

D(blue) = (-1, -1), D(red) = (1, 0), D(yellow) = (0, 1)

Interpolate:

$$q = \sum_i \alpha_i p^i \Rightarrow g(q) = \sum_i \alpha_i g(p^i)$$

$$\therefore g(q) = q + \underbrace{\sum_i \alpha_i D(p^i)}_{D(q)}$$

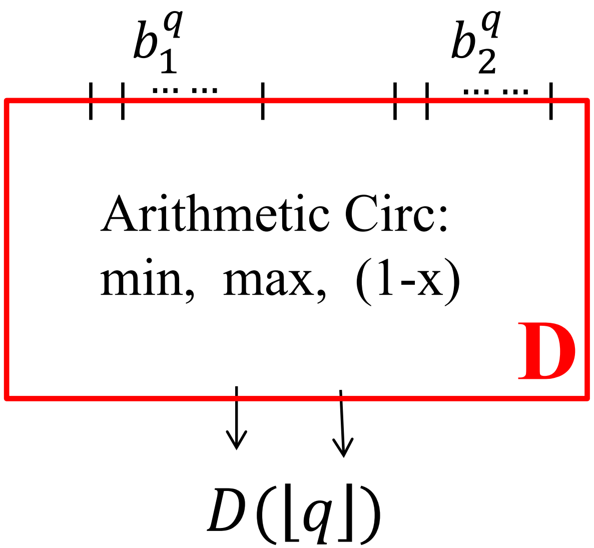
Claim: q is a fixed point of g
iff $D(q) = 0$
iff q is center of a trichromatic triangle

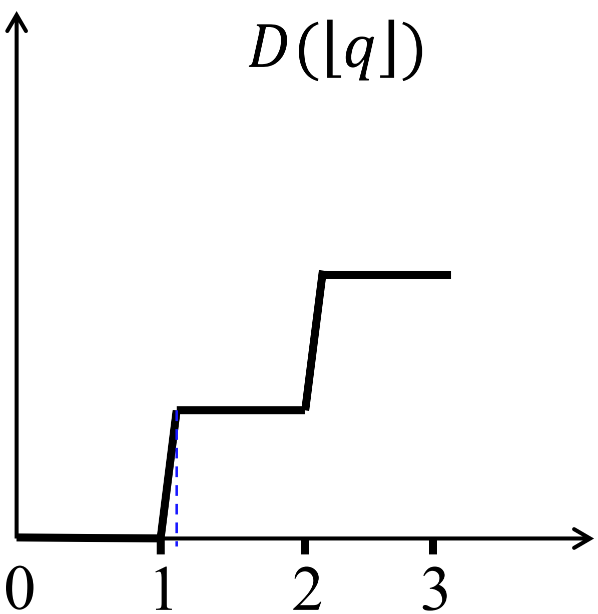
To evaluate $D(p^1)$
need bits($\lfloor q \rfloor$)

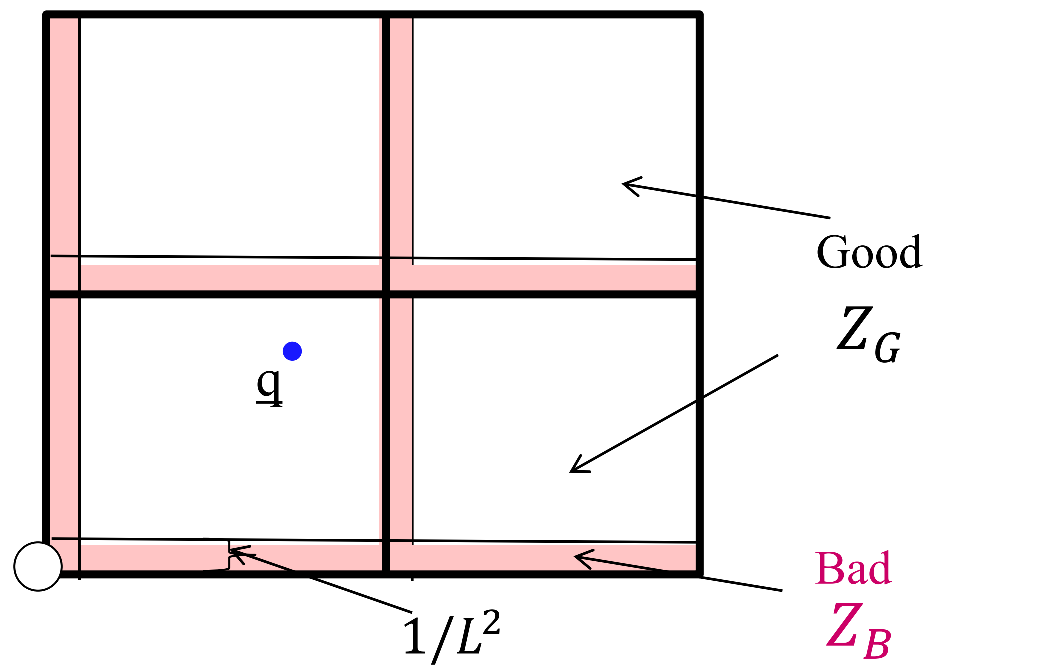
Floor is a discontinuous function !

Computing bits of $\lfloor \mathbf{q} \rfloor$

L: large number







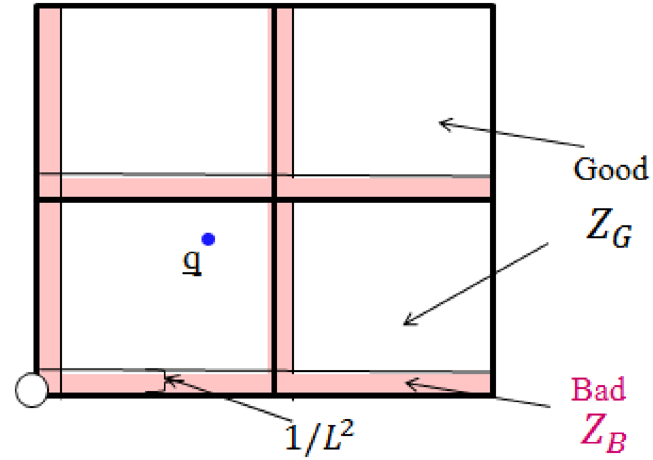
$b^q = \text{bits}(\lfloor q \rfloor)$ if in $Z_G \Rightarrow$ correct $D(\lfloor q \rfloor)$

Else $\mathbf{0} \leq b_{1i}^q,\ b_{2i}^q \leq \mathbf{1}$,
$i = 1, .., n$

Junk!
$D(\lfloor q \rfloor) \in [-1, 1]^2$

Computing bits($\lfloor q_d \rfloor$), $d = 1,2$ when $q \in Z_G$

1. Set $a = q_d$
2. For $i = n-1, \ldots, 0$
   1. $b_{di} = \min\{\max\{(a - 2^i) * L^2,\ 0\}, 1\}$
   2. $a = a - (b_{di} * 2^i)$



$q = \text{int} + \text{frac}$

$\text{frac} \in \left[\frac{1}{L^2}, 1\right]$

Claim: (i) At the start of any iteration if $a \leq 2^i \Rightarrow b_{di} = 0$

if $a \geq 2^i + \frac{1}{L^2} \Rightarrow b_{di} = 1.$

(ii) If $a \in Z_G$ at the start of an iteration, then at the end of it too $a \in Z_G$

Requires O(n) sized arithmetic circuit.

Sampling Lemma (CDT'06)

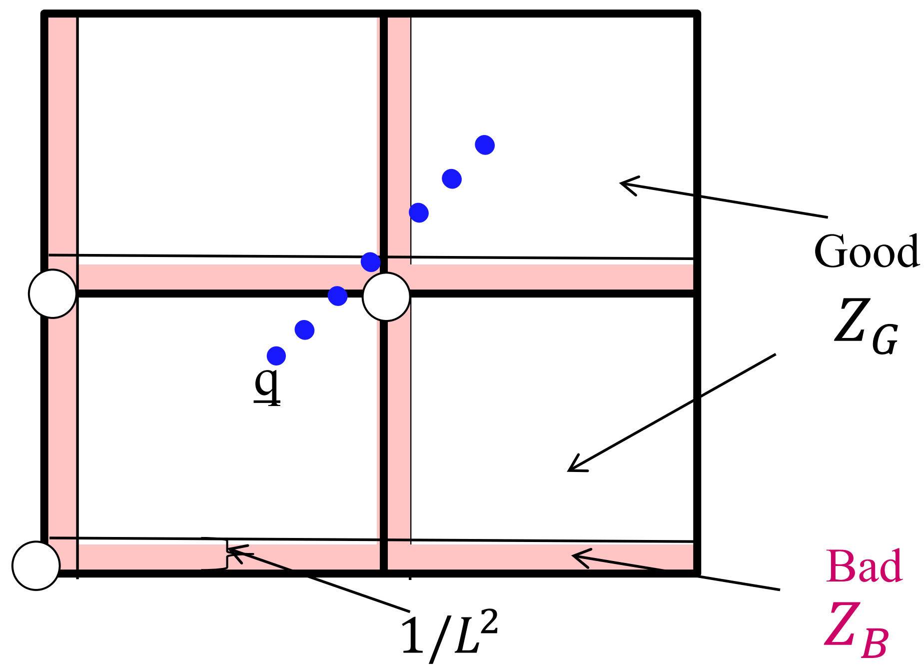
Given q $\in [0, 2^n - 1]^2$,

$$q^{\boldsymbol{k}} = q + \frac{k-1}{L}, 1 \leq k \leq 16$$

$r^k = D(\lfloor q^{\boldsymbol{k}} \rfloor)$ if $q^{\boldsymbol{k}}$ in Z_G,

else $r^k \in [-1, 1]^2$.

If $r = \sum_k r^k = 0$ then q $\in$ trichromatic square.



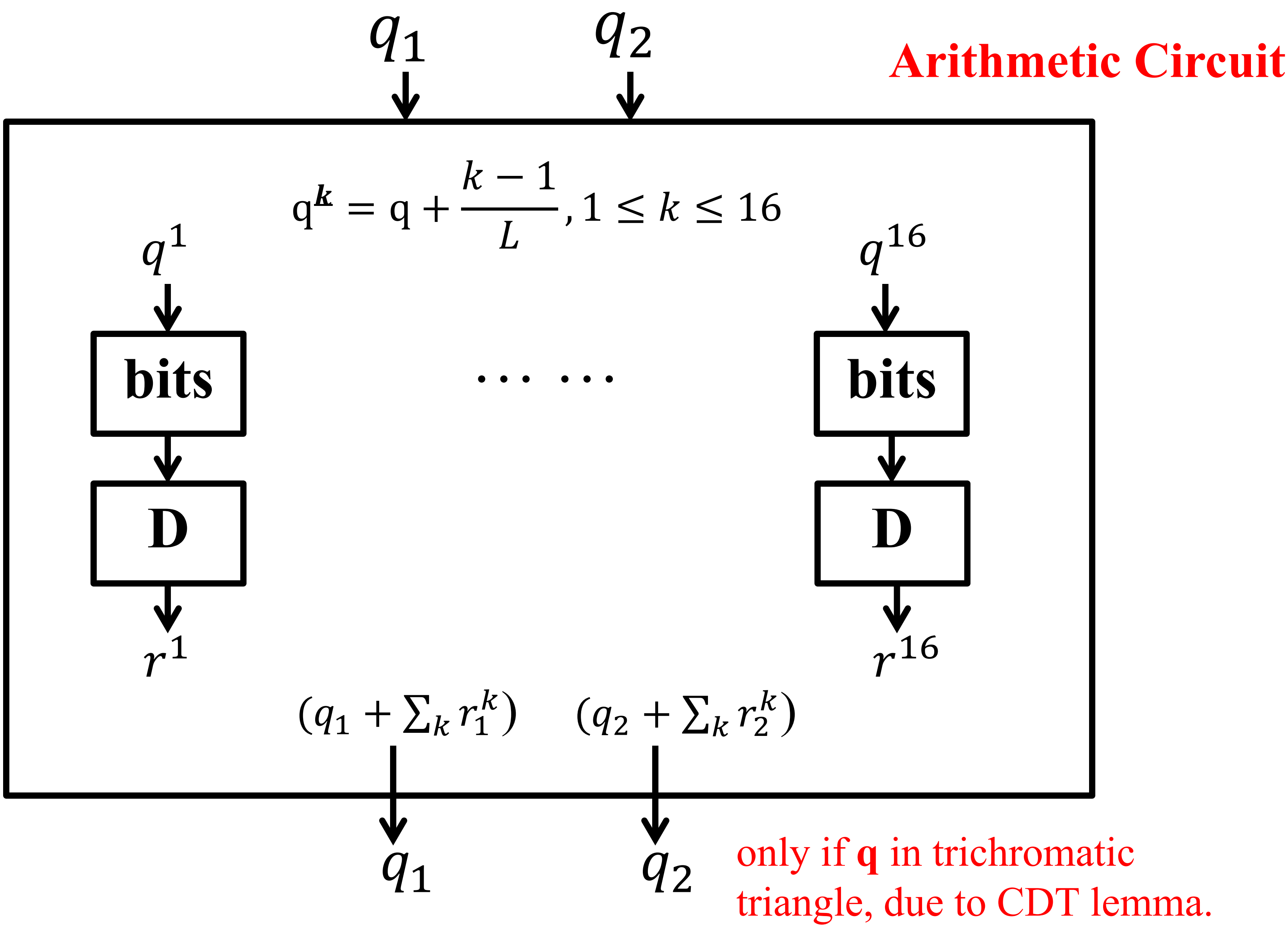
Proof: At most two r^ks in Z_B, generating junk $\in [-2, 2]^2$.

In $\{r^k, \; k \in Z_G\}$: If no yellow ⇒ all are either (1, 0) or (-1, -1) ⇒ $r_2 \leq -1$, or $r_1 \geq 1$

If no red ⇒ only (0, 1) or (-1,-1) ⇒ $r_1 \leq -1$, or $r_2 \geq 1$

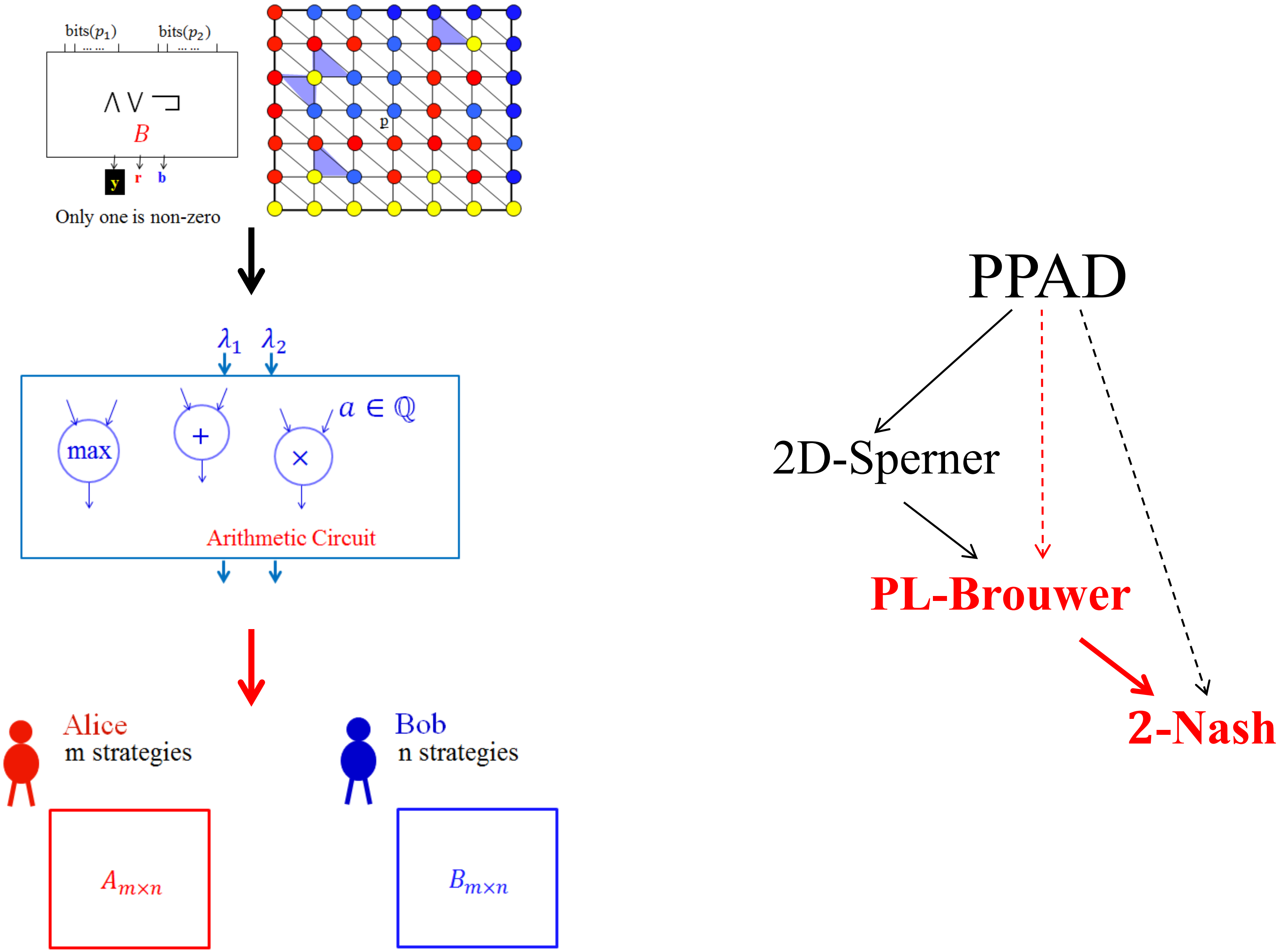
If no blue ⇒ only (1, 0) or (0, 1) ⇒ $\max\{r_1, r_2\} \geq 1$

2D-Sperner → PL-Brouwer



2D-Sperner → PL-Brouwer

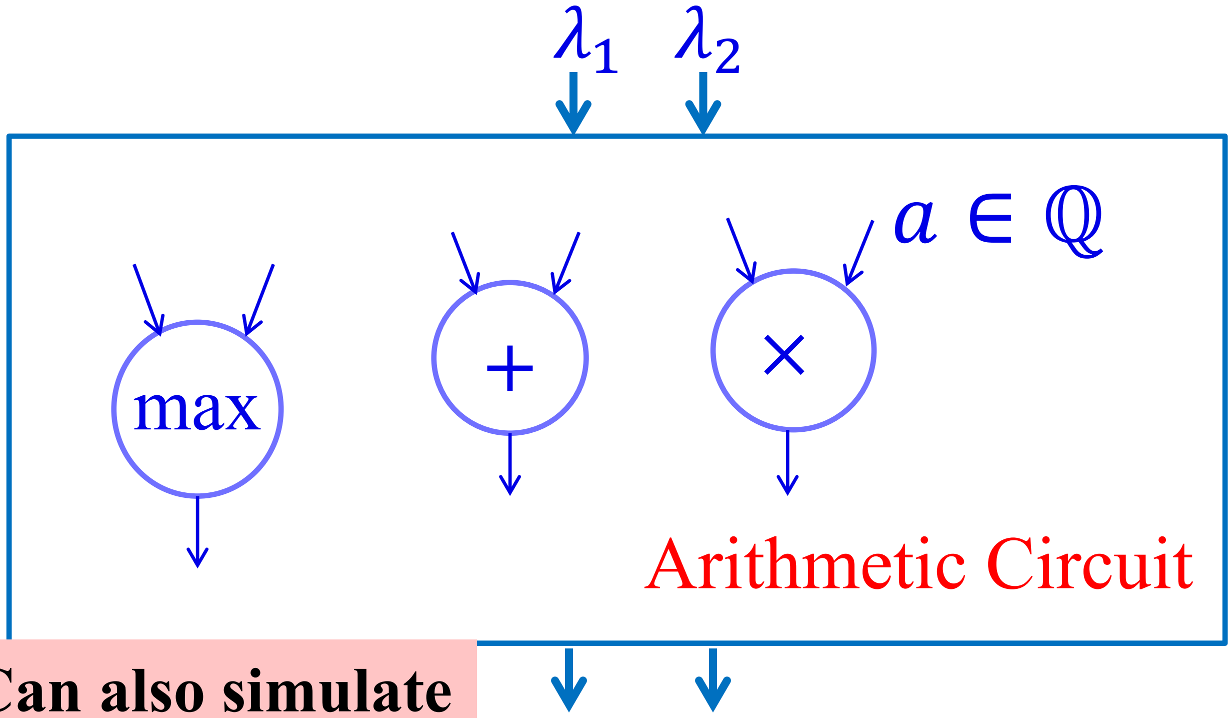
Theorem 1: 2D-Sperner is polynomial-time reducible to (2D) PL-Brouwer.


bits(p_1)
bits(p_2)
∧ ∨ ¬
B
y
r
b
Only one is non-zero
p
λ_1
λ_2
max
+
×
$a \in \mathbb{Q}$
Arithmetic Circuit
Alice
m strategies
Bob
n strategies
$A_{m\times n}$
$B_{m\times n}$
PPAD
2D-Sperner
PL-Brouwer
2-Nash

$2D$-PL-Brouwer

$F: [0,1]^2 \to [0,1]^2$

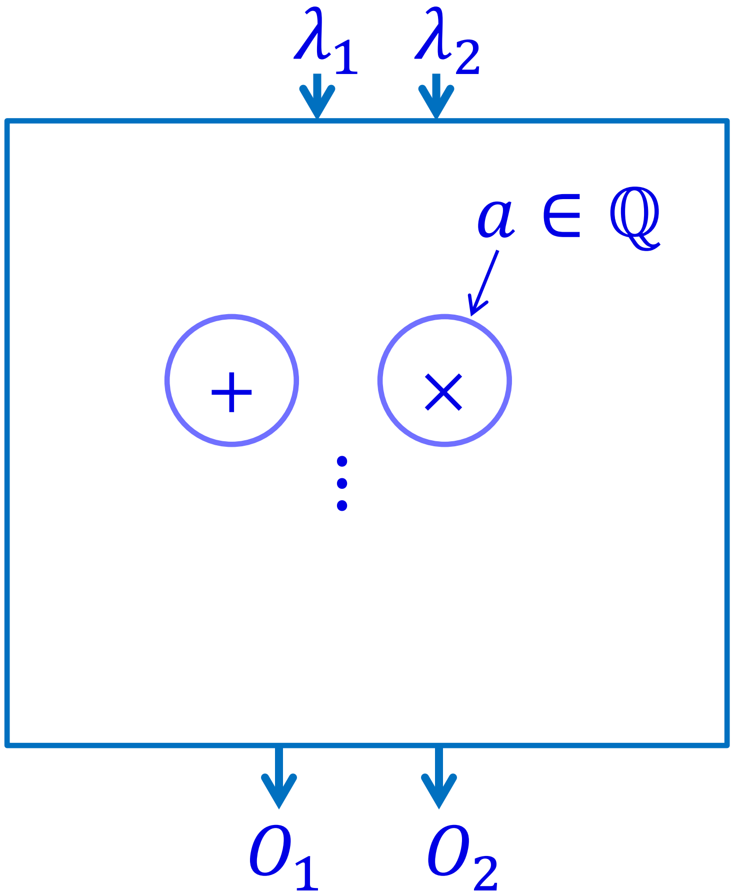
Find a fixed Point of F: $x \in [0,1]^k$ s.t. $F(x) = x$



Can also simulate min and minus

No Max Gate

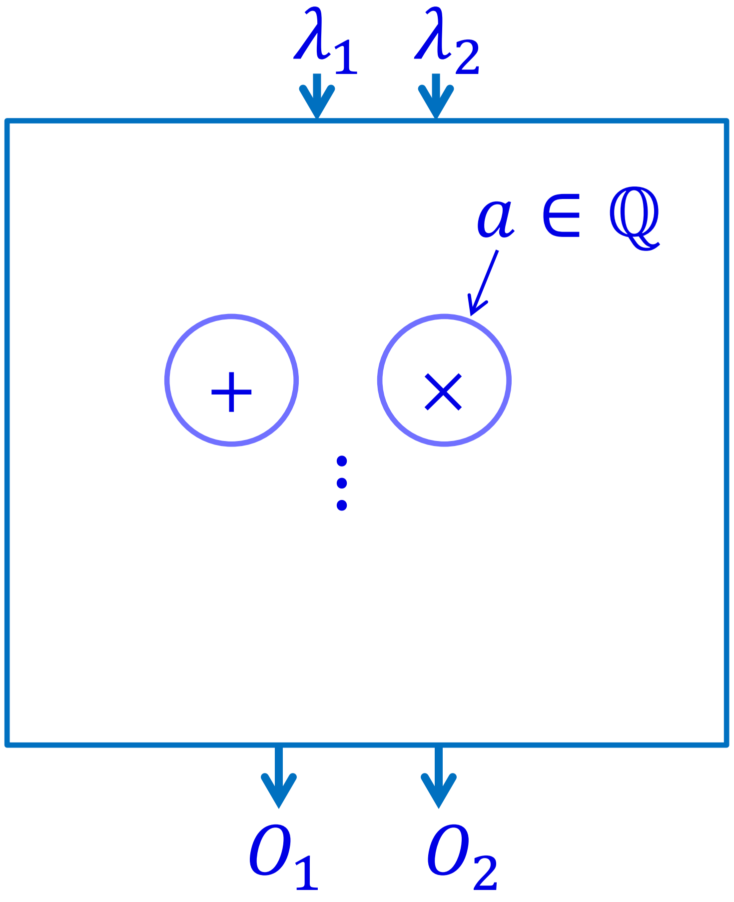
Example on board.

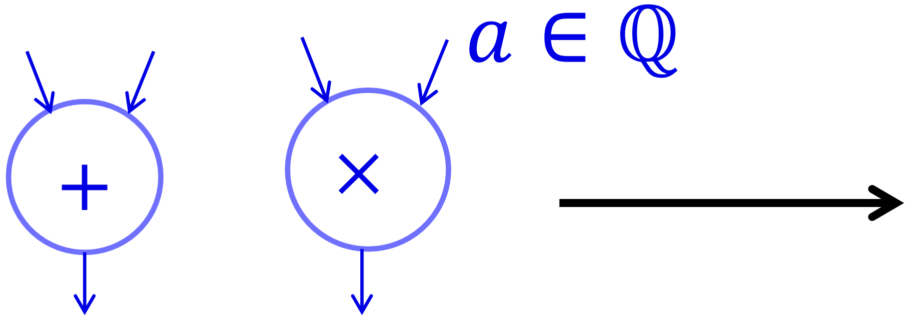


No Max Gate

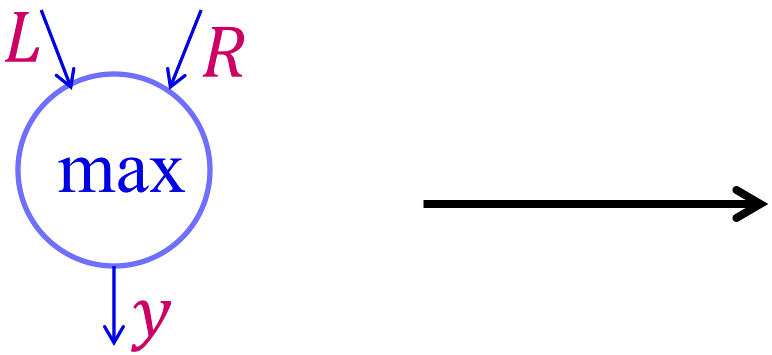
$$O_1 := a_1\lambda_1 + b_1\lambda_2 + c_1$$
$$O_2 := a_2\lambda_1 + b_2\lambda_2 + c_2$$

Fixed-point: $O_1 = \lambda_1$
$O_2 = \lambda_2$

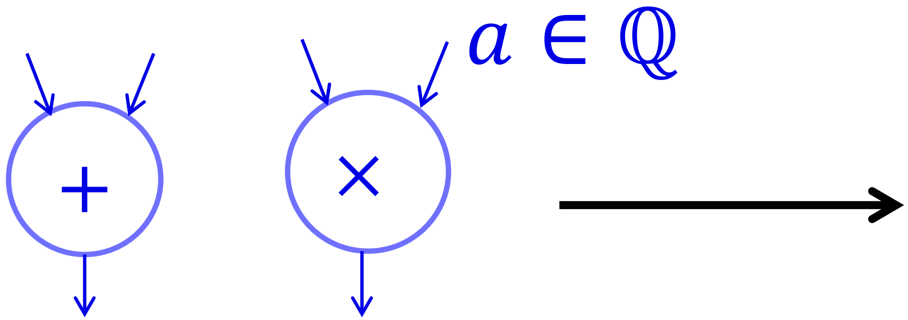




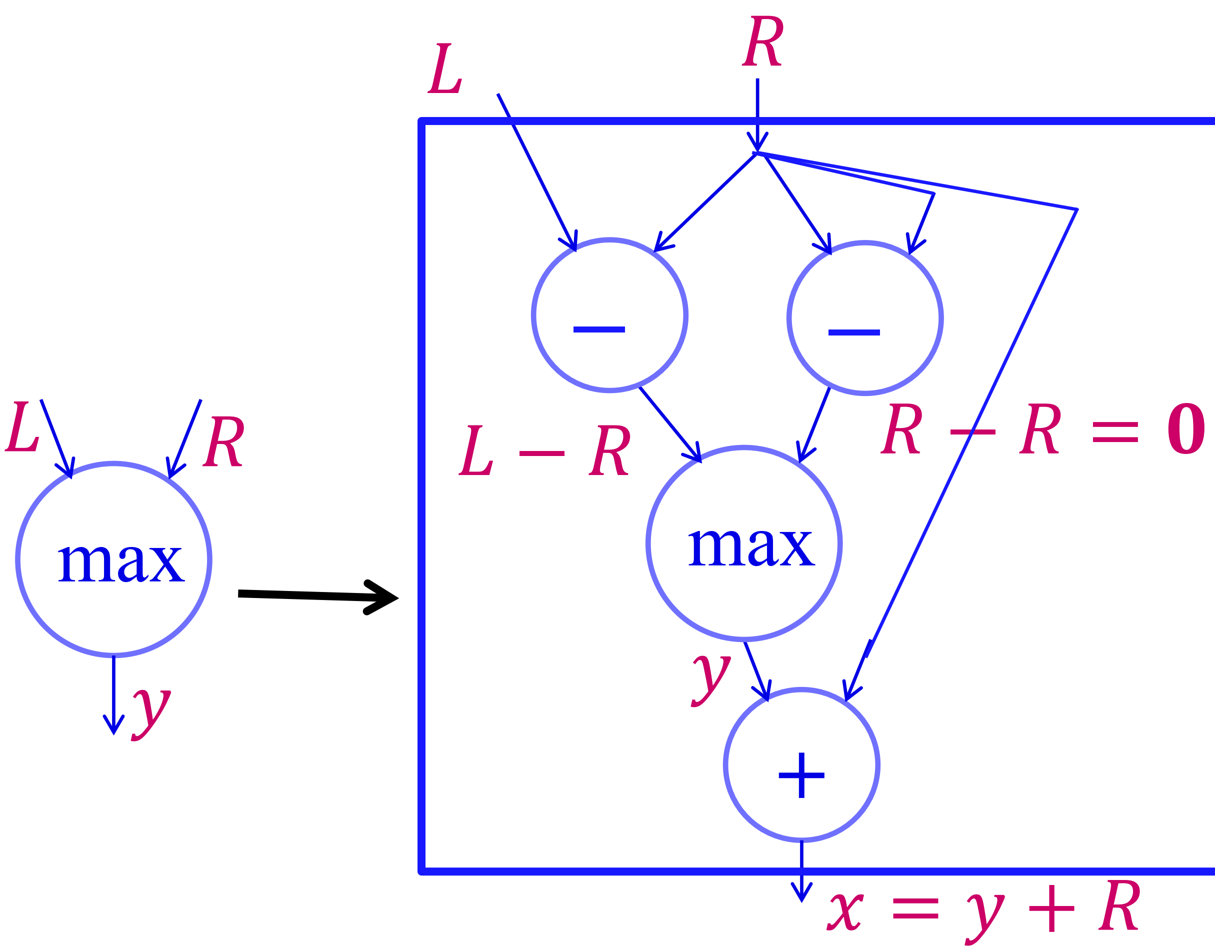
Linear
Expression

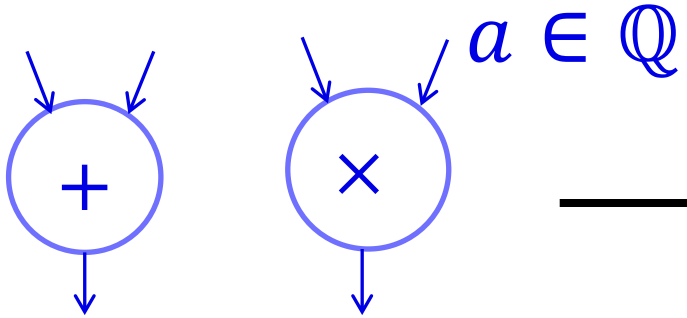


$$y \geq L, \qquad y \geq R$$
$$(y - L)(y - R) = 0$$

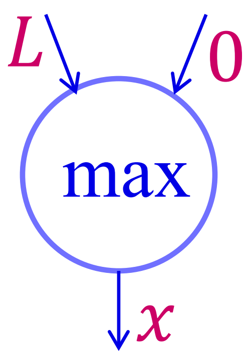


Linear
Expression

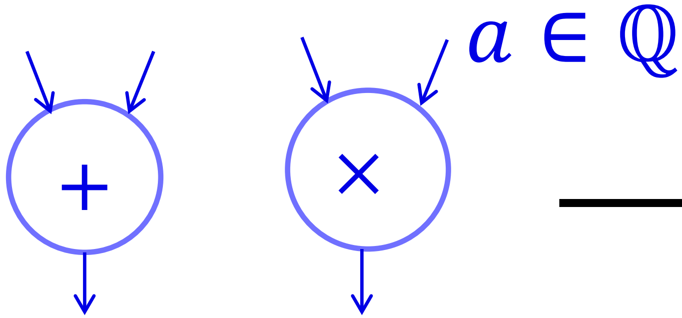




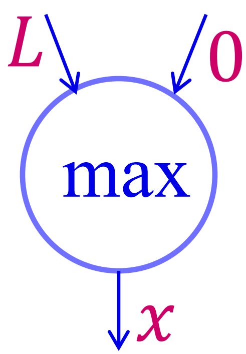
Linear Expression



$$x \geq L, \qquad x \geq 0$$
$$(x - L)x = 0$$



→ Linear Expression

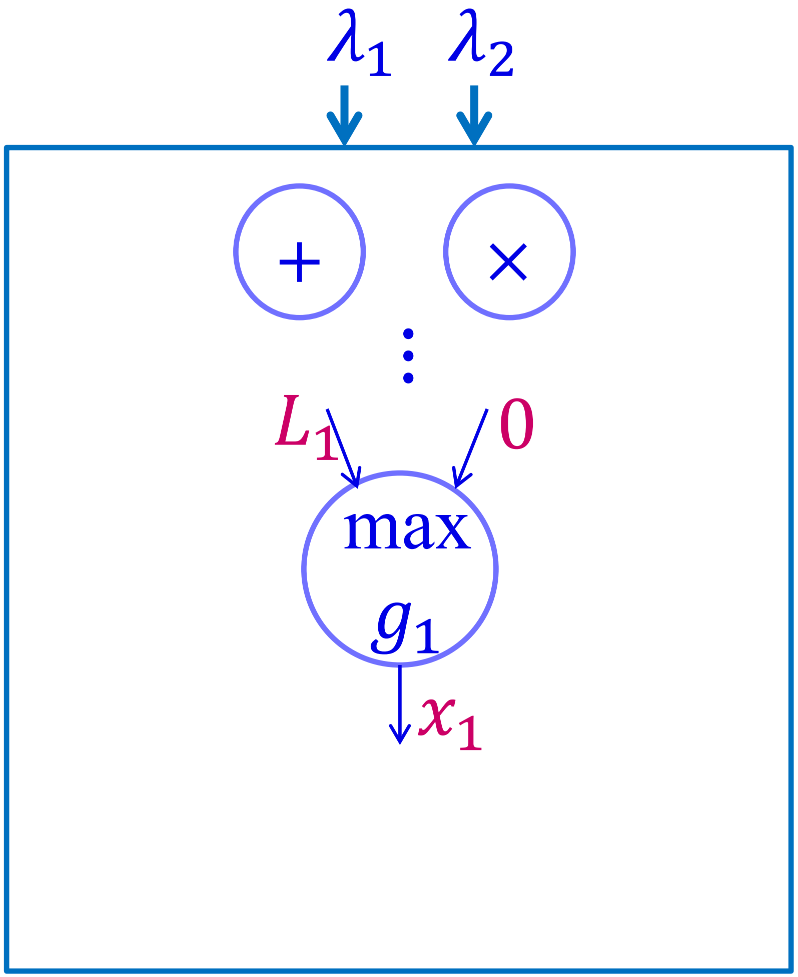


→ $x \geq 0 \perp x \geq L$

Modeling *max* Gates

- Ordering among *max* gates: $g_1, \ldots, g_n$

- L_1- Linear in λ_1 & λ_2
  $x_1 \geq 0 \perp x_1 \geq L_1$

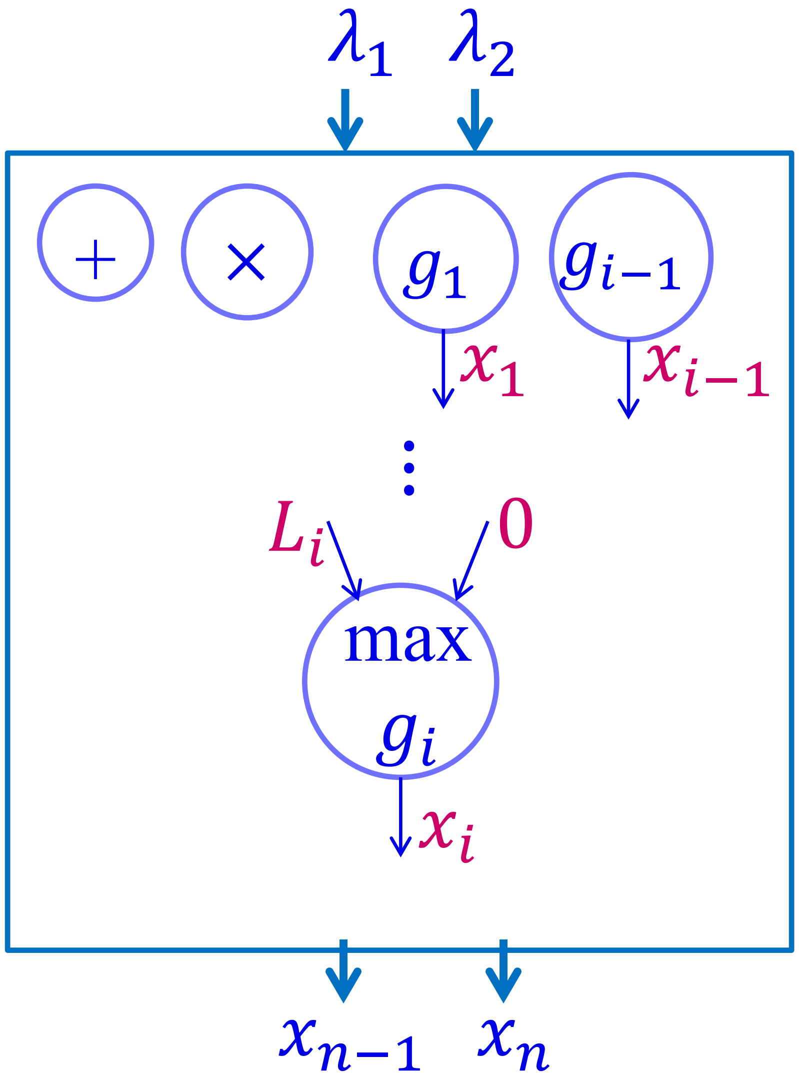


Modeling *max* Gates

- L_i- Linear in $x_1, \ldots, x_{i-1}, \lambda_1, \lambda_2$

$$x_i \geq 0 \perp x_i \geq L_i$$

n: #max gates



Lemma 1: Given (λ_1, λ_2), $(x_1, \ldots, x_n)$ satisfy the following

$$S\colon\ x_i \geq 0 \ \perp \ x_i \geq L_i(x_1, \ldots, x_{i-1}, \lambda_1, \lambda_2), \forall i \leq n$$

$$\Updownarrow$$

$$F(\lambda_1, \lambda_2) = (x_{n-1}, x_n)$$

Proof: $(\Rightarrow)$ By construction, x_i has to be the output of i^{th} max gate when input to the circuit is (λ_1, λ_2).

$(\Leftarrow)$ Evaluate the circuit of F at given (λ_1, λ_2).
Set x_i to output of i^{th} max gate, then $(x_1, \ldots, x_n)$ has to satisfy *S*, because two of the inputs of this max gate are ‘0’ and expression L_i.

$\boldsymbol{S}$: $x_i \geq 0 \perp x_i \geq L_i(x_1, \ldots, x_{i-1}, \lambda_1, \lambda_2), \forall i \leq n$

$\downarrow$

$\underline{x} \geq 0 \perp A\underline{x} \geq \lambda_1 \underline{d} + \lambda_2 \underline{e} + \underline{b}$

Fixed Point $\Rightarrow \lambda_1 = x_{n-1}, \lambda_2 = x_n$

$A \in R^{n \times n}$ lower-triangular with 1s on diagonal

$\equiv$

$\underline{x} \geq 0 \perp A\underline{x} \geq x_{n-1}\underline{d} + x_n \underline{e} + \underline{b}$

$A' = A - [\underline{0}, \ldots, \underline{0}, \underline{d}, \underline{e}]$

$\equiv$

LCP: $\underline{x} \geq 0 \perp A'\underline{x} \geq \underline{b}$

LCP = Linear Complemetarity Problem

Lemma 2: $\underline{x}$ is a solution of the LCP iff it is a solution of $\boldsymbol{S}$,
(then using Lemma 1) iff (x_{n-1}, x_n) is a fixed point.

LCP → Game

LCP: $\underline{x} \geq 0 \quad \perp \quad A'\underline{x} \geq \underline{b}$

$$Z = \begin{bmatrix} -A' & \underline{b} + \underline{1} \\ \underline{0} & 1 \end{bmatrix}$$

$Symmetric\ Game\colon (Z, Z^T)$

Recall: Nash Eq. of (A, B)

$\underline{y} \in \Delta_n \qquad \underline{x} \in \Delta_m$

Complementarity:

$$\forall i, x_i \geq 0 \perp \left(A\underline{y}\right)_i \leq \alpha$$

$$\forall j, y_j \geq 0 \perp \left(\underline{x}^T B\right)_j \leq \beta$$

$$\Updownarrow$$

1. $(\underline{x}, \underline{y})$ is a NE
2. α and β are the payoffs

Symmetric Nash Eq. of (Z, Z^T)

$m = n$ $\quad\quad$ $\underline{x} \in \Delta_m$

$\underline{x} = \underline{y}$

Complementarity:

$$\forall i, x_i \geq 0 \perp (Z\underline{x})_i \leq \alpha$$

$$\forall i, x_i \geq 0 \perp (\underline{x}^T Z^T)_i \leq \beta$$

$\equiv$

$\Updownarrow$

1. $(\underline{x}, \underline{x})$ is a Symm. NE
2. α and β are the payoffs

Symmetric Nash Eq. of (Z, Z^T)

$$\underline{x} \in \Delta_m$$

Complementarity: $\forall i, x_i \geq 0 \perp (Z\underline{x})_i \leq \alpha$

$$\Updownarrow$$

1. $(\underline{x}, \underline{x})$ is a Symm. NE
2. α is the payoff to both

LCP → Game

LCP: $\underline{x} \geq 0 \quad \perp \quad A'\underline{x} \geq \underline{b}$

Claim: A' is strictly semi-monotone.

$$Z = \begin{bmatrix} -A' & \underline{b} + \underline{1} \\ \underline{0} & 1 \end{bmatrix}$$

Symmetric Game: (Z, Z^T)

Theorem 2: If $(\underline{x}, t)$ is a symmetric NE (both play the same), then $\underline{x}/_t$ is the LCP solution.

LCP → Game (Proof)

LCP: $\underline{x} \geq 0 \;\perp\; A'\underline{x} \geq \underline{b}$

$$Z = \begin{bmatrix} -A' & \underline{b} + \underline{1} \\ \underline{0} & 1 \end{bmatrix} \begin{bmatrix} \underline{x} \\ t \end{bmatrix}$$

Game: (Z, Z^T)

$(\underline{x}, t)$ be a symmetric Nash

$$\forall i, \qquad x_i \geq 0 \;\perp\; \left(-A'\underline{x}\right)_i + b_i t + t \leq \alpha$$

$$t \geq 0 \;\perp\; t \leq \alpha$$

LCP → Game (Proof)

LCP: $\underline{x} \geq 0 \quad \perp \quad A'\underline{x} \geq \underline{b}$

$$\forall i,\ x_i \geq 0 \quad \perp \quad \left(-A'\underline{x}\right)_i + b_i t + t \leq \alpha$$
$$t \geq 0 \quad \perp \quad t \leq \alpha$$

Claim 1: If $t > 0$ then $\frac{x}{t}$ is a soln of the LCP.

Proof:

$$t > 0 \;\Rightarrow\; t = \alpha \;\Rightarrow\; \left(A'\left(\frac{\boldsymbol{x}}{t}\right)\right)_i \geq b_i$$

LCP → Game (Proof)

LCP: $\underline{x} \geq 0 \quad \perp \quad A'\underline{x} \geq \underline{b}$

$$\forall i, \;\; x_i \geq 0 \quad \perp \quad (-A'\underline{x})_i + b_i t + t \leq \alpha$$
$$t \geq 0 \quad \perp \quad t \leq \alpha$$

Claim 2: $t > 0$, if A′ is strictly semi-monotone.

$\underline{x} \geq 0, \underline{x} \neq 0$ then
$\exists i, x_i > 0, (A'x)_i > 0$

Proof: $t = 0 \Rightarrow \alpha \geq 0$

$x_i > 0$ and $(-A'x)_i < 0 \leq \alpha$. Contradiction!

Main Result

Theorem 3: Given circuit of function F, construct game (Z, Z^T). $(\underline{x}, \mathrm{t})$ is a symmetric NE of this game iff $\left(\frac{x_{n-1}}{t}, \frac{x_n}{t}\right)$ is a fixed-point of F.

(Recall) Theorem 1: $\left(\frac{x_{n-1}}{t}, \frac{x_n}{t}\right)$ is a fixed-point of F iff it is in trichromatic triangle of the 2D-Sperner problem.

Main Theorem: 2D-Sperner is polynomial-time reducible to symmetric NE in symmetric 2-player game.

Symmetric 2-Nash $\longrightarrow$ 2-Nash

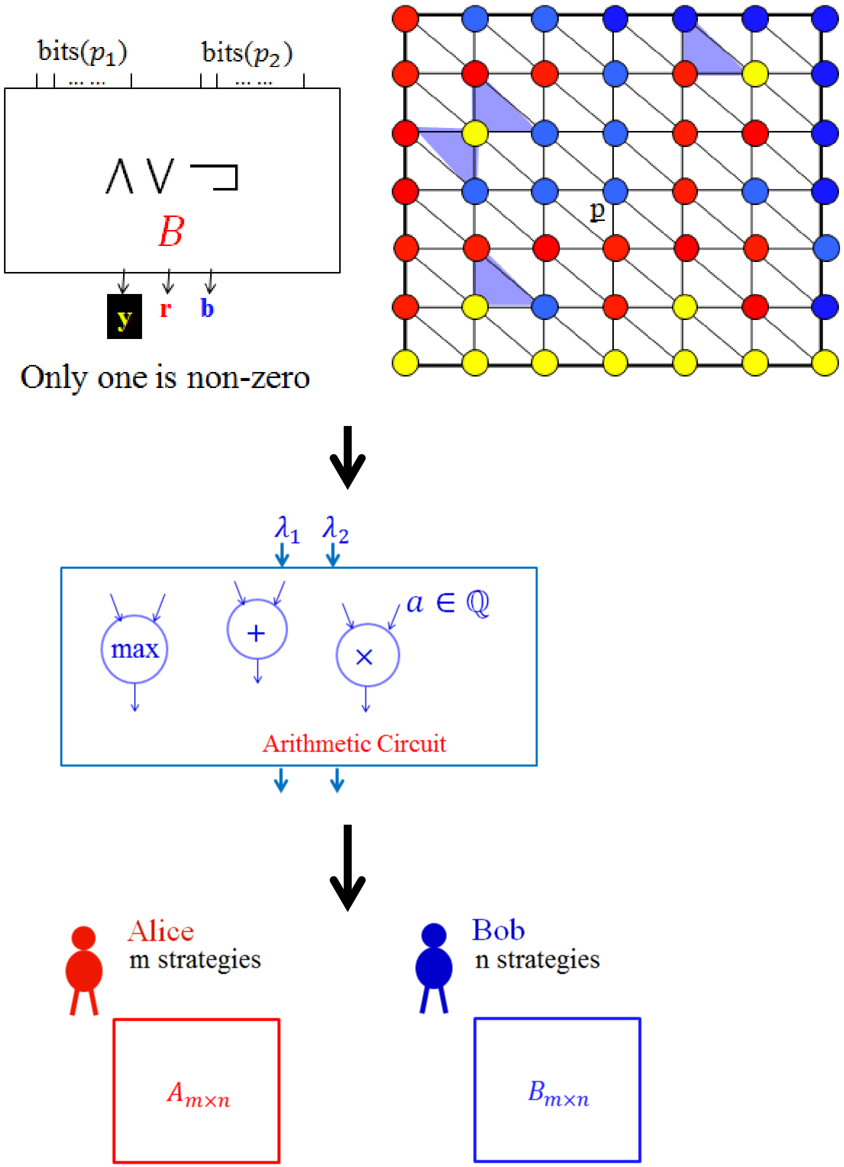
$$(Z, Z^T) \longrightarrow (I, Z)$$

$$(\underline{x}, \underline{x}) \longleftarrow (\underline{x}, \underline{y})$$

"Imitation Games"

(McLennan and Tourky'10)

Q.E.D.!


bits(p_1)
bits(p_2)
∧ ∨ ¬
B
y r b
Only one is non-zero
p
λ_1 λ_2
max
+
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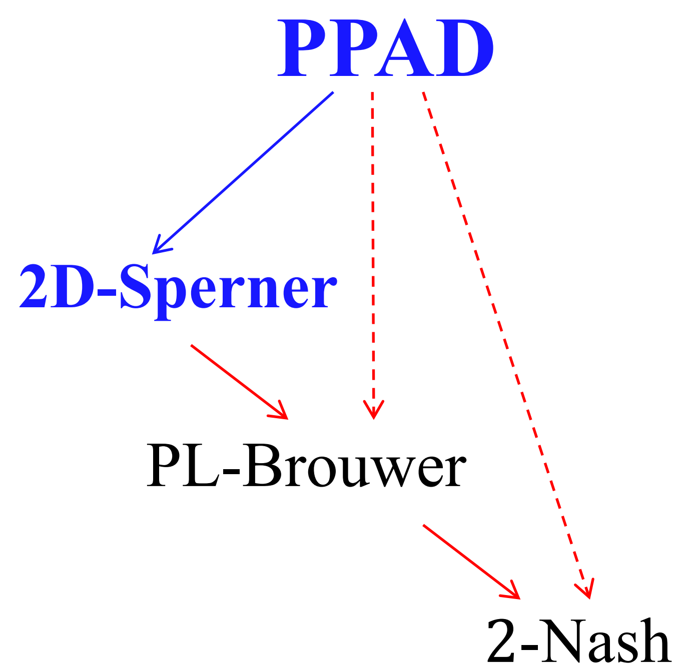
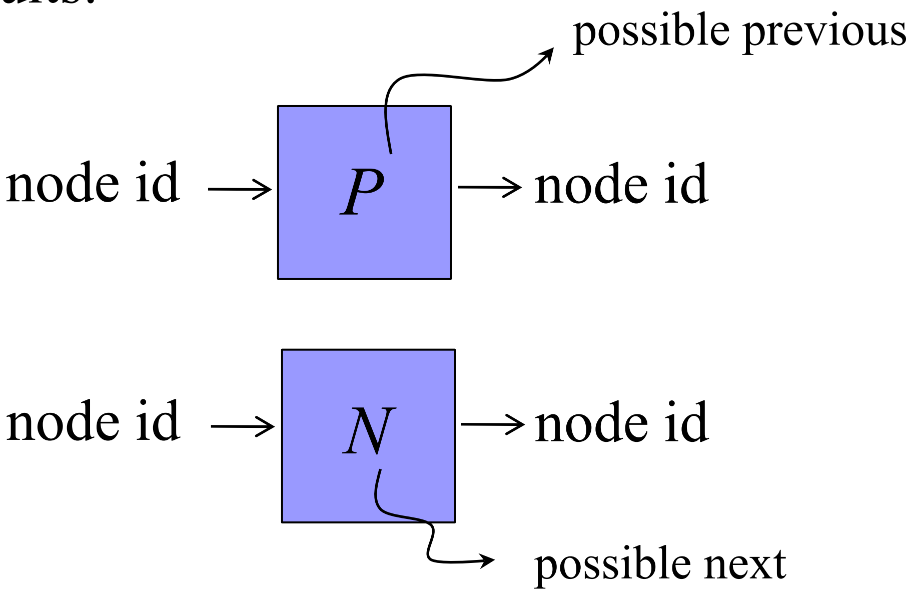
$a \in \mathbb{Q}$
Arithmetic Circuit
Alice
m strategies
Bob
n strategies
$A_{m \times n}$
$B_{m \times n}$

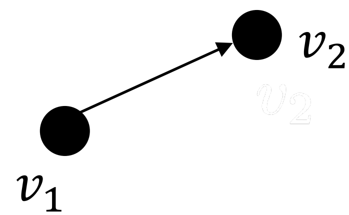

PPAD
2D-Sperner
PL-Brouwer
2-Nash

Recall: The PPAD Class

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by two circuits:

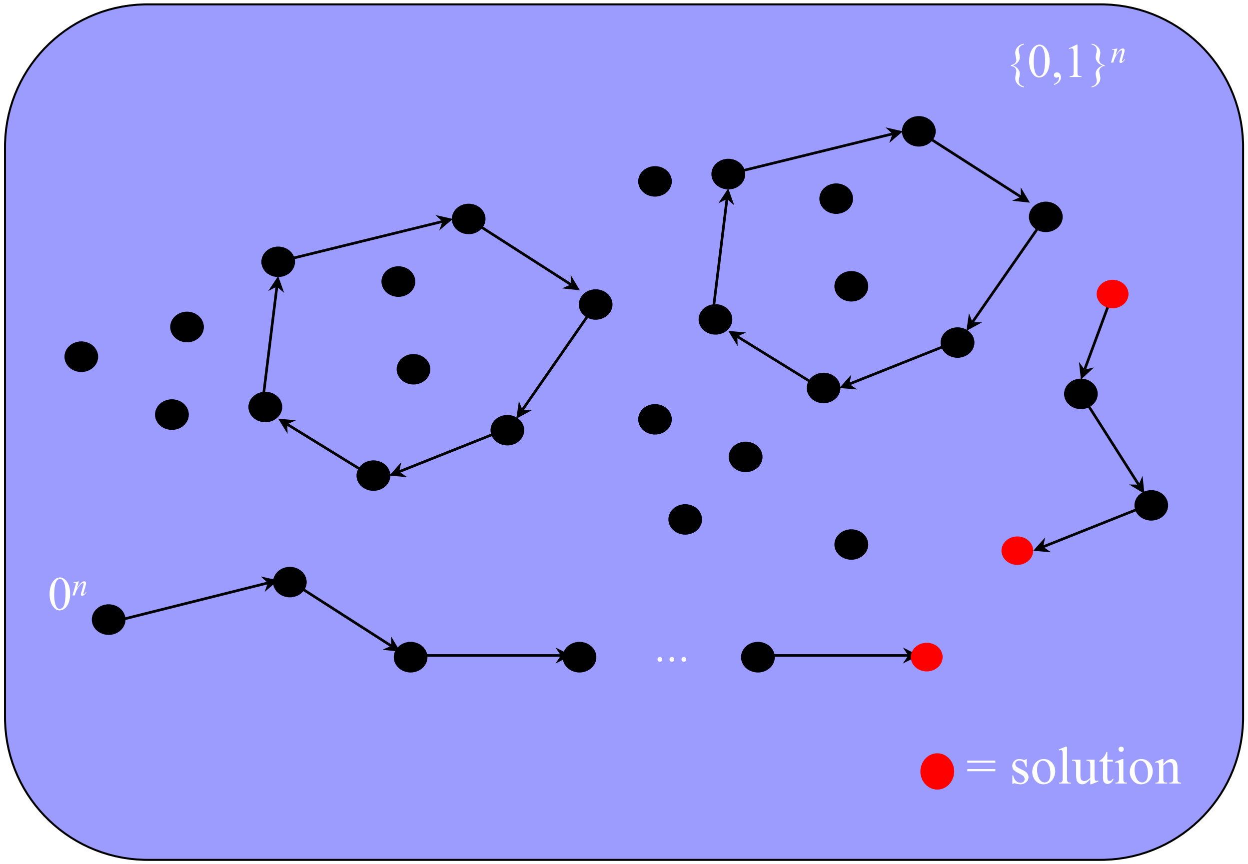


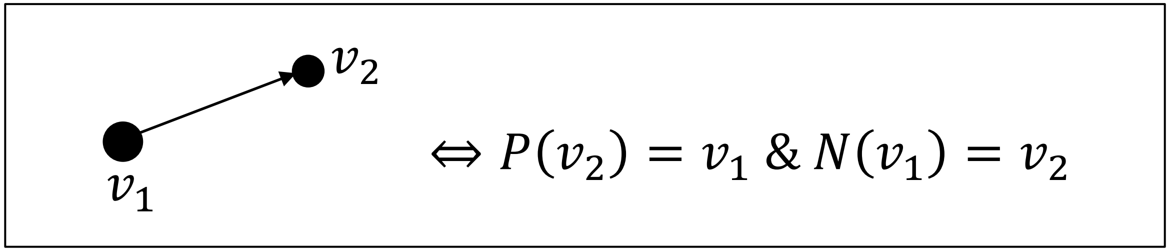
$$P(v_2) = v_1 \;\&\; N(v_1) = v_2$$



END OF A LINE: Given P and N, and unbalanced node 0^n, find another unbalanced node.

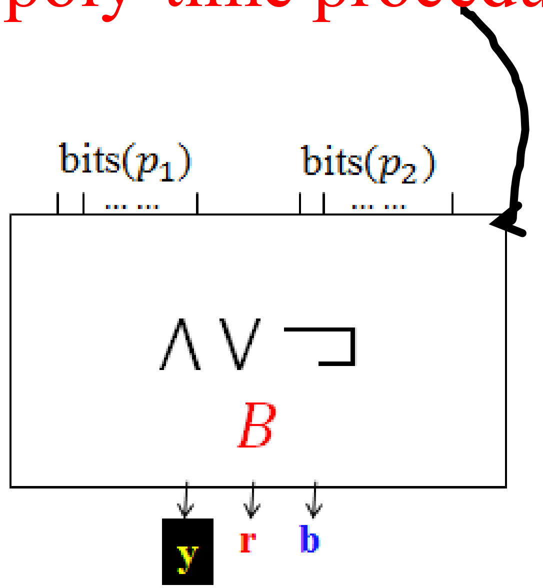
Recall: END OF A LINE



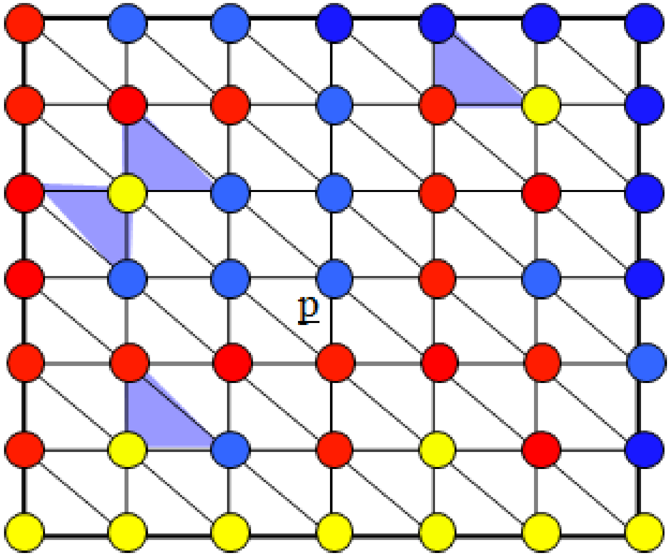


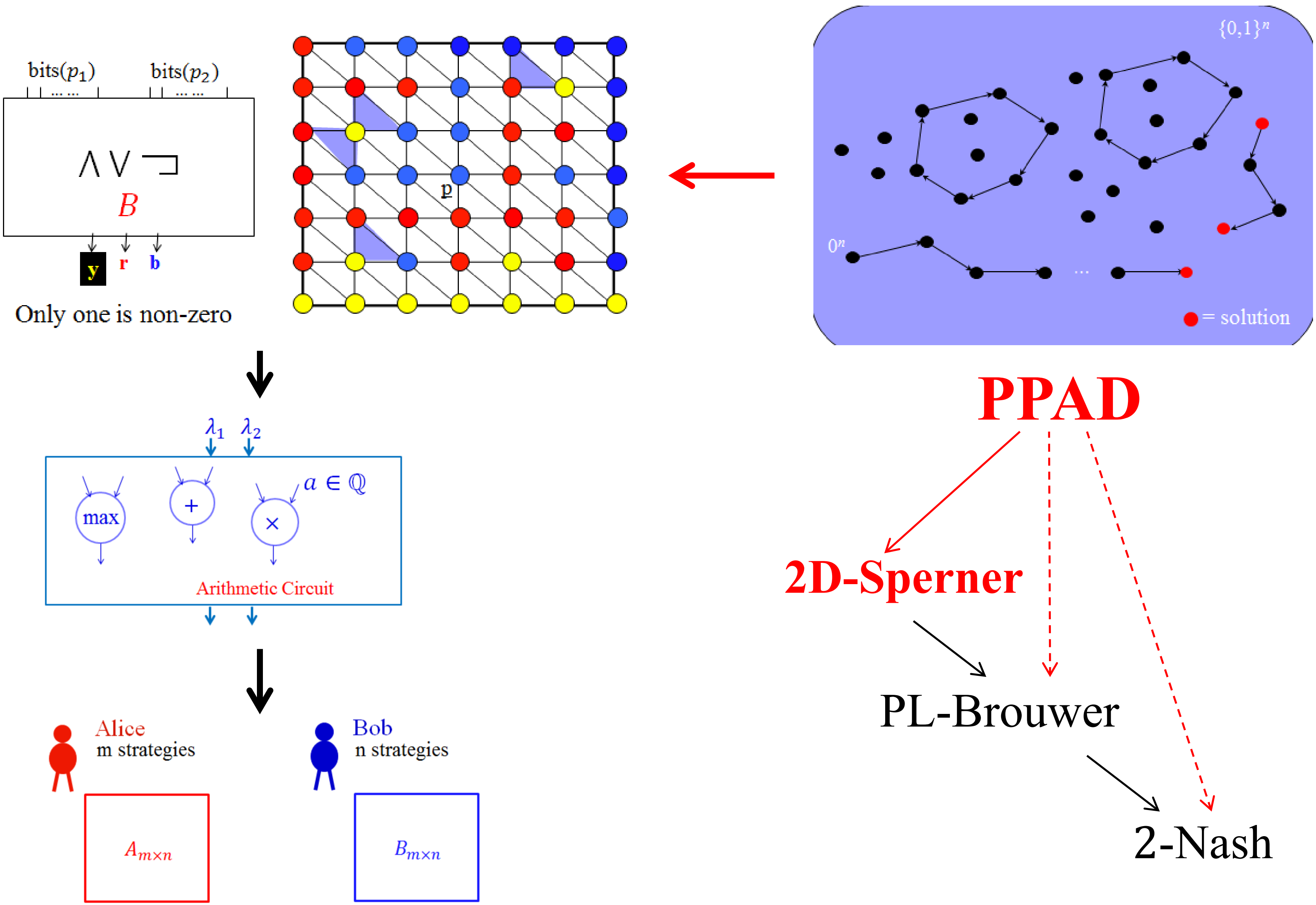
Recall: 2D-Sperner

Can also be thought of
as poly-time procedure



Only one is non-zero





bits(p_1) bits(p_2)
… … … …
$\wedge \vee \neg$
B
y r b
Only one is non-zero
p
$\{0,1\}^n$
0^n
…
= solution
PPAD
2D-Sperner
PL-Brouwer
2-Nash
λ_1 λ_2
max
+
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$a \in \mathbb{Q}$
Arithmetic Circuit
Alice
m strategies
Bob
n strategies
$A_{m\times n}$
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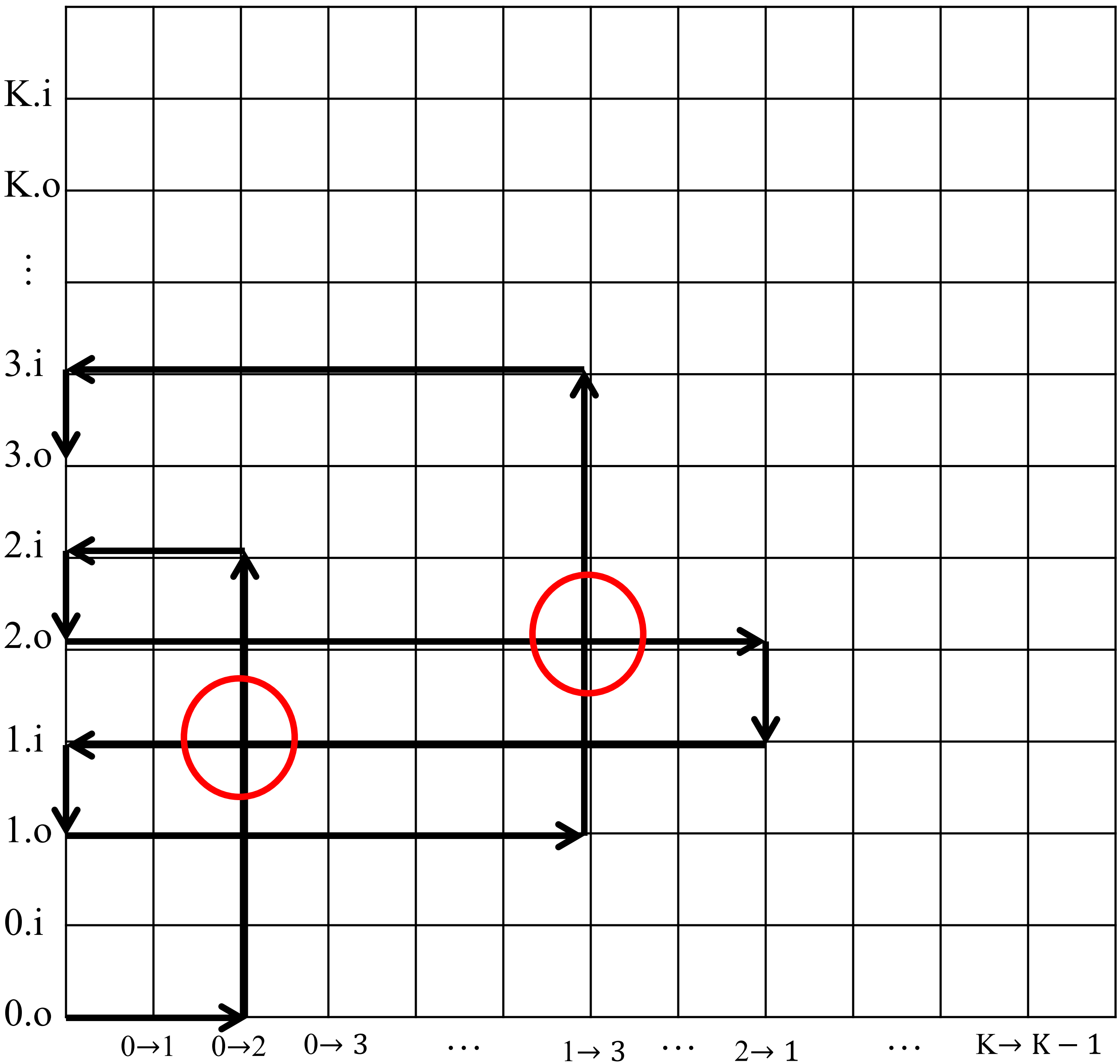
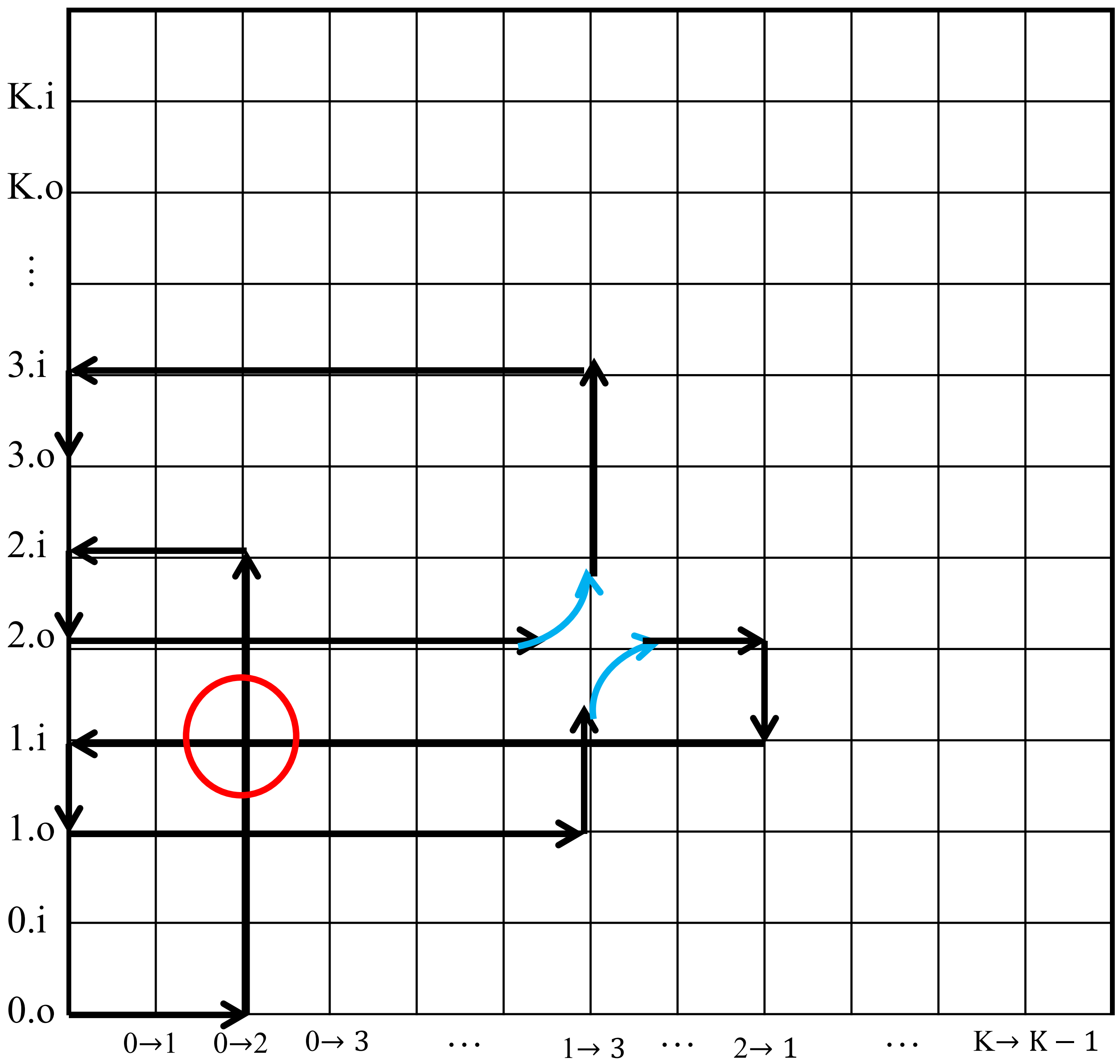


$K=2^n-1$

$v_0 \rightarrow v_2 \rightarrow v_1 \rightarrow v_3$

How to Avoid crossings?



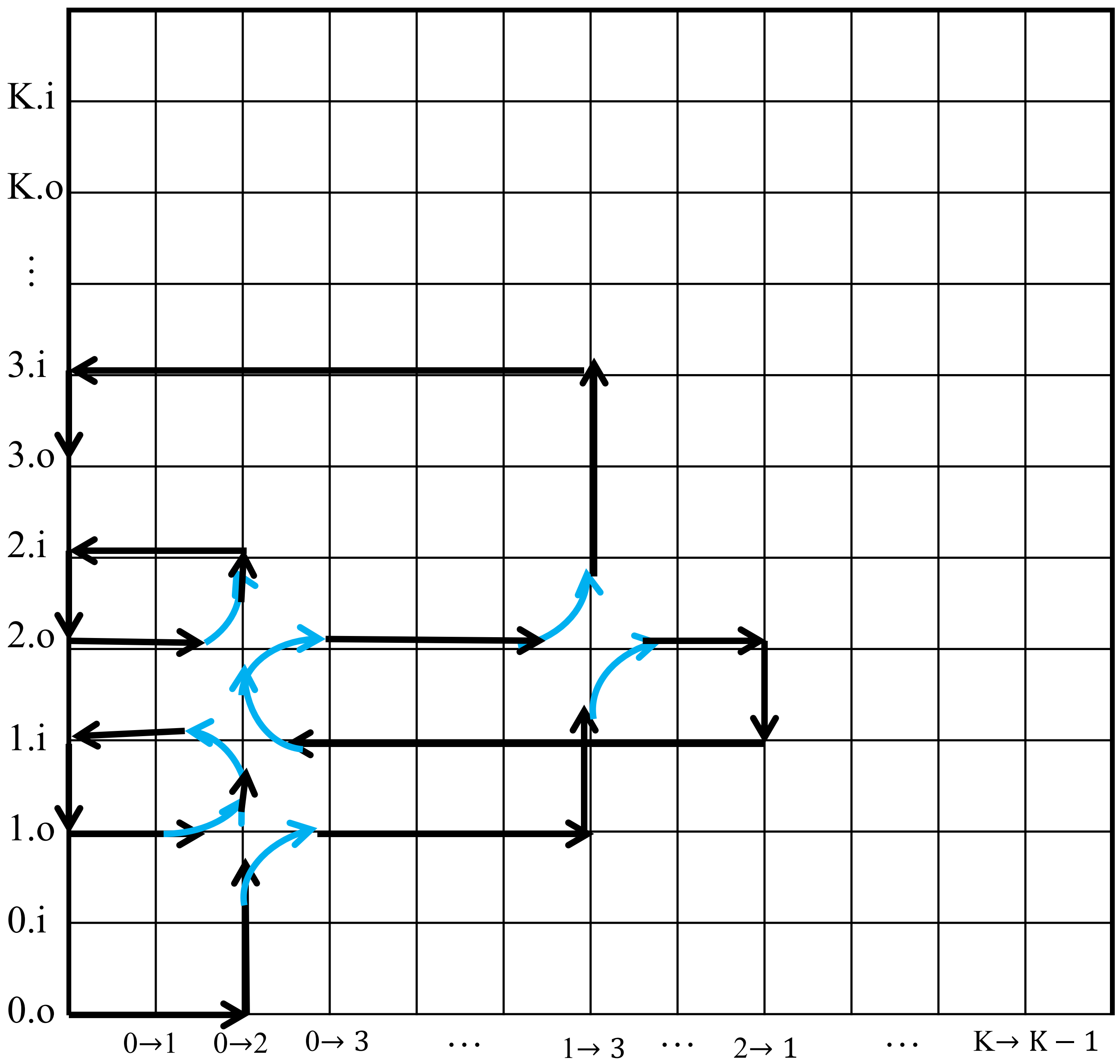
K=$2^n - 1$

$v_0 \rightarrow v_2$

$\rightarrow v_1$

$\rightarrow v_3$

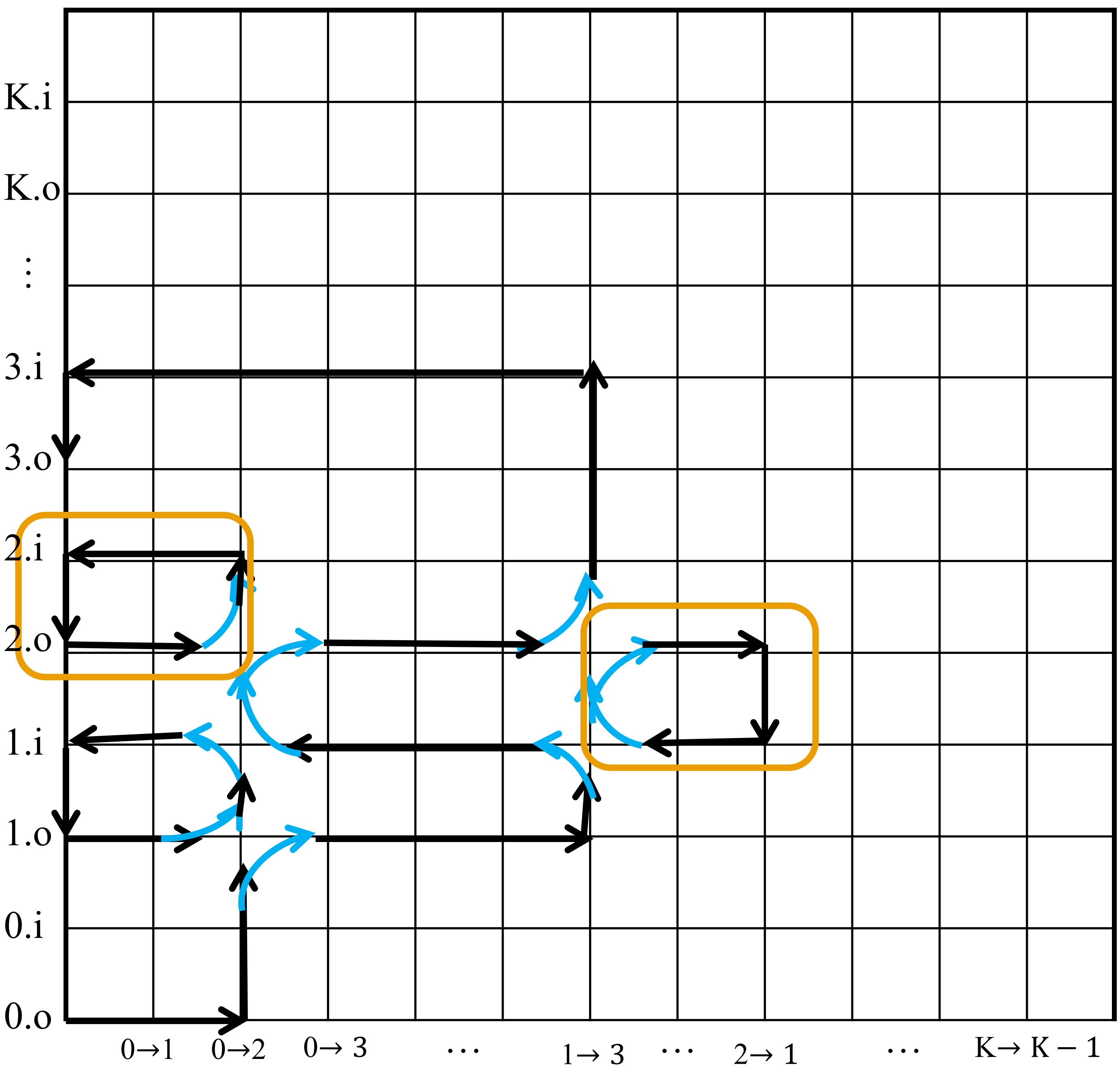
How to Avoid crossings?



$K=2^n-1$

$v_0 \to v_2$
$\to v_1$
$\to v_3$

How to Avoid crossings?

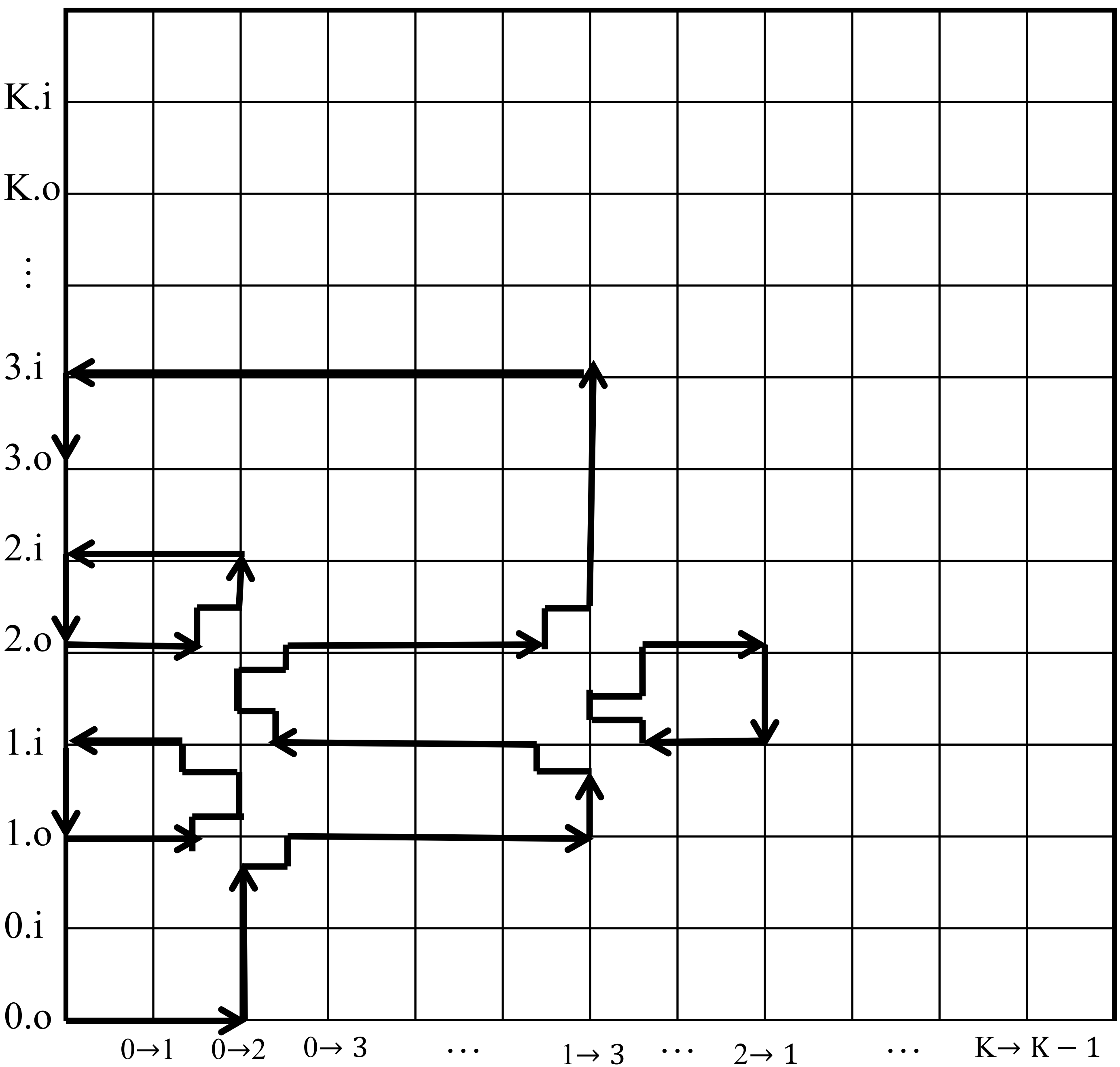


$K=2^n-1$

$v_0 \to v_2 \to v_1 \to v_3$

Extra cycles

But no extra End-points



$K=2^n - 1$

$v_0 \to v_2 \to v_1 \to v_3$

Extra cycles

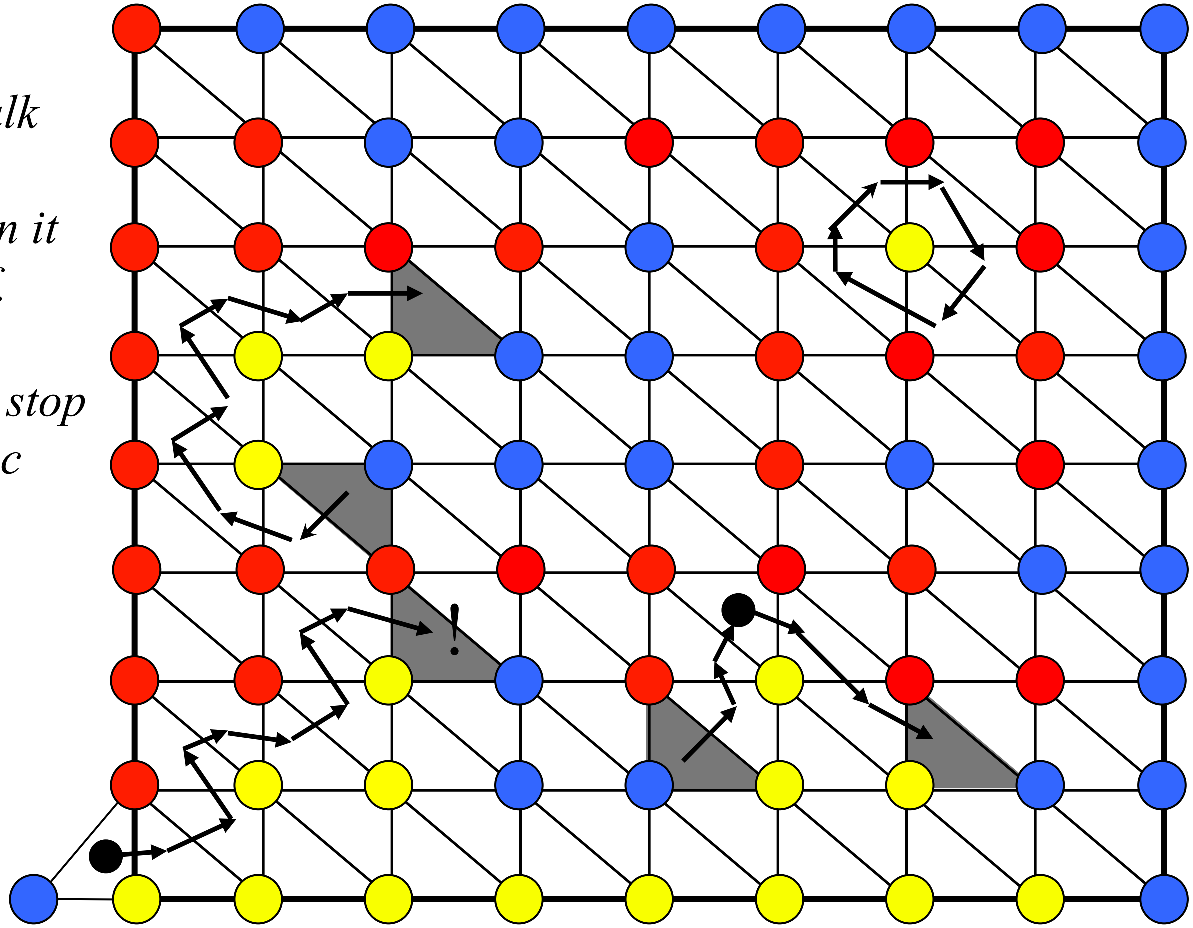
But no extra End-points

Coloring?

Proof of Sperner's Lemma

Claim**:** *The walk cannot exit the square, nor can it loop into itself.*

Hence, it must stop at tri-chromatic triangle…



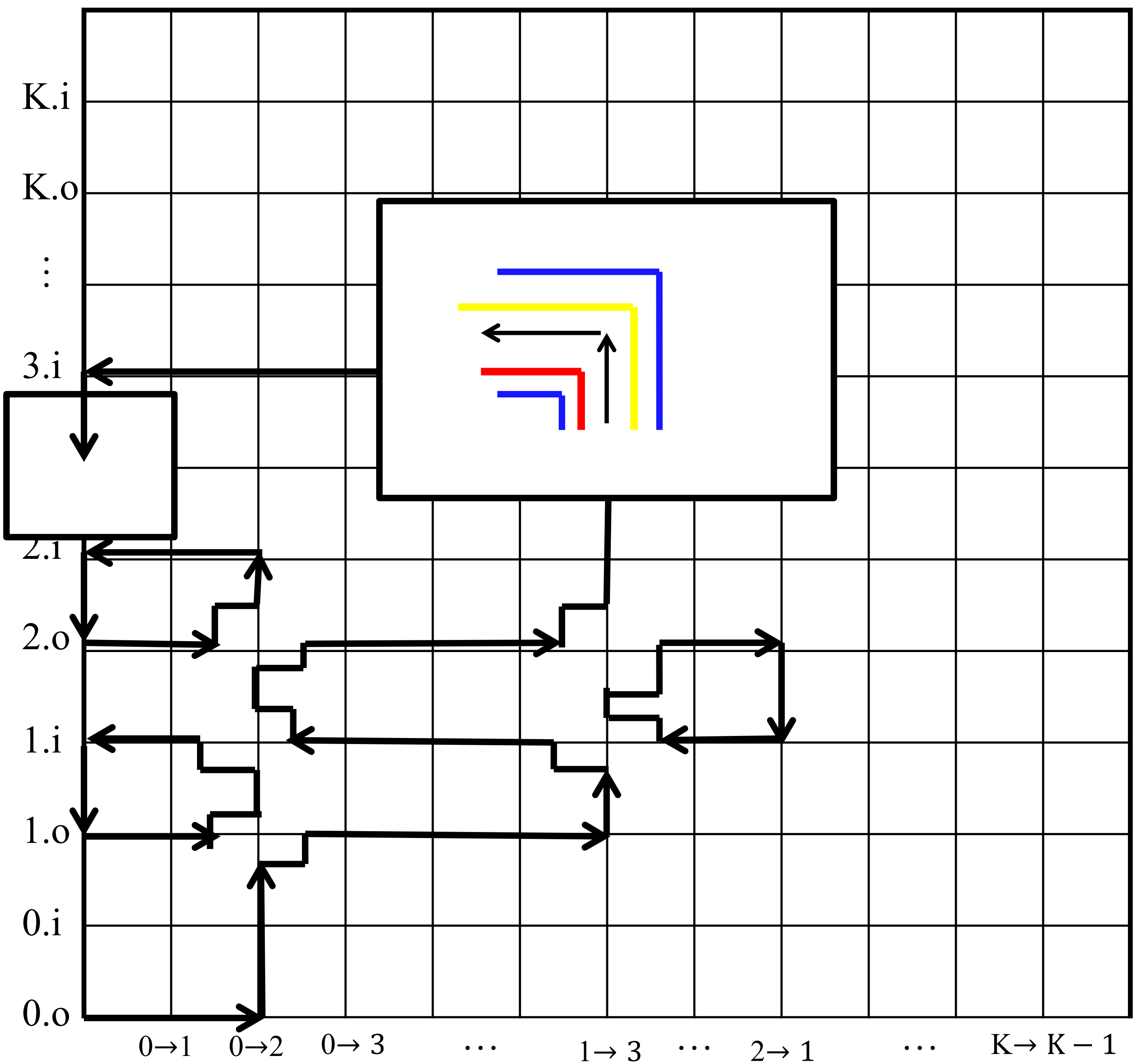
Rule:

Cross R – Y

Keeping

R on left

[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.



$K=2^n - 1$

$v_0 \to v_2$

$\to v_1$

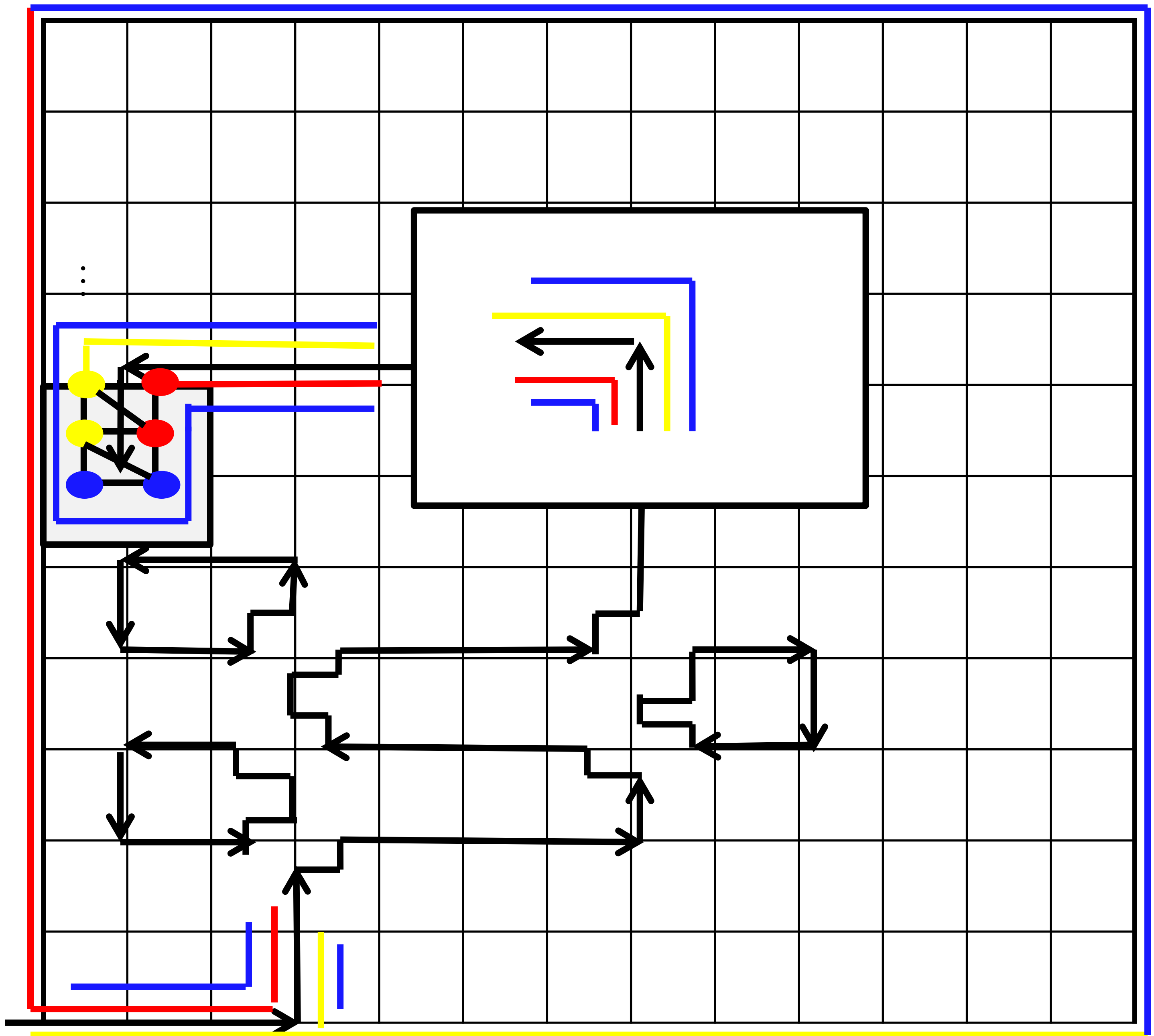
$\to v_3$

Extra cycles

But no extra
End-points

Coloring?

Constant
Blow-up



$K = 2^n - 1$

$v_0 \rightarrow v_2$

$\rightarrow v_1$

$\rightarrow v_3$

Extra cycles

But no extra End-points

Coloring?

Constant Blow-up

Poly-time Procedure for Coloring?

- Every row uniquely correspond to a vertex in/out
- Every column uniquely correspond to $v_i \rightarrow v_j$
  - For every grid point of the initial grid, can use circuit N and P of PPAD to check if a line is passing through it and in what direction.
- Even further subdivision in constantly many cells is predetermined.
- What about crossings?
  - Again checking and uncrossing are locally possible.