Lecture 8
Games and Nash Equilibrium

CS 598RM

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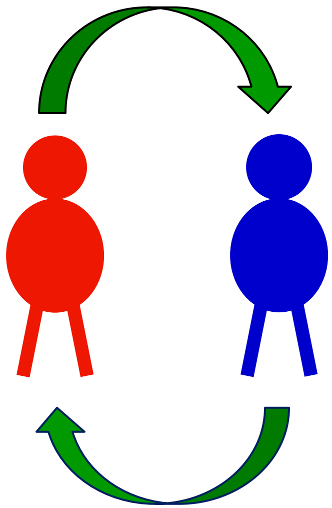
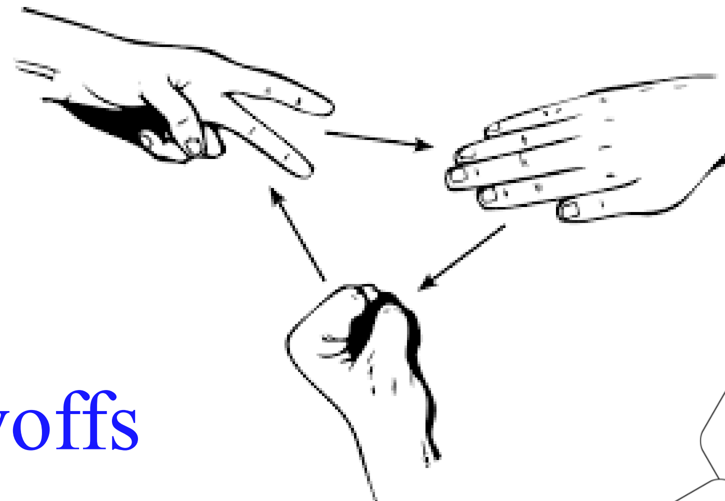
Games

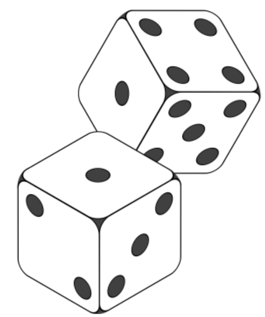


Payoffs



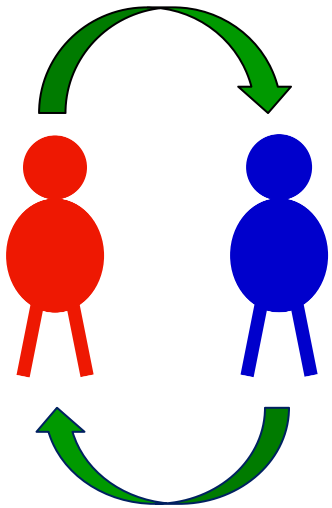
Players

Strategies

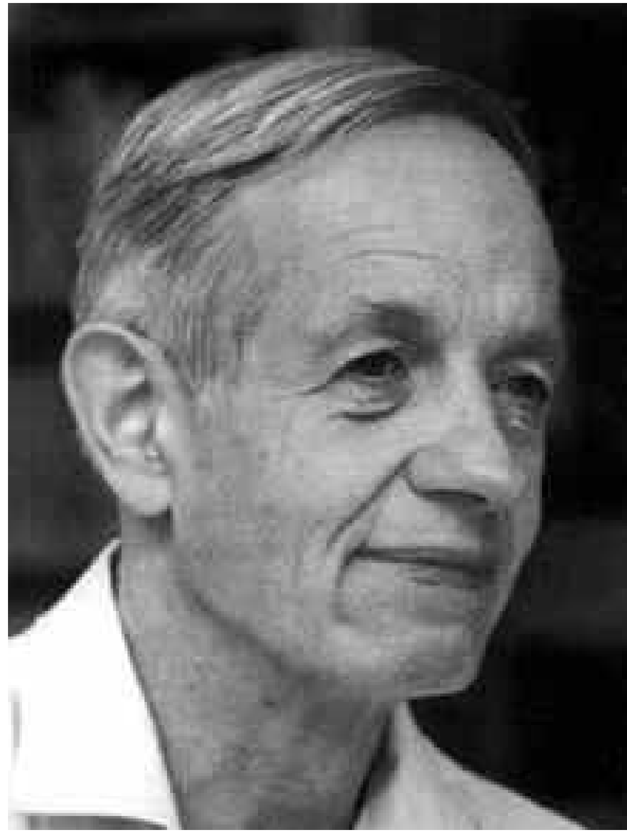


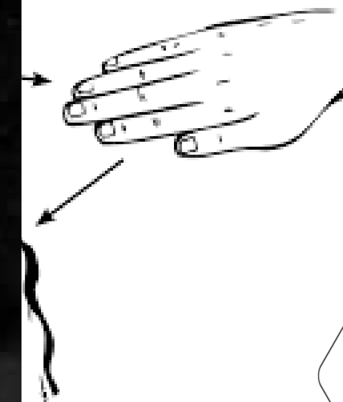
Randomize!

Games



Players





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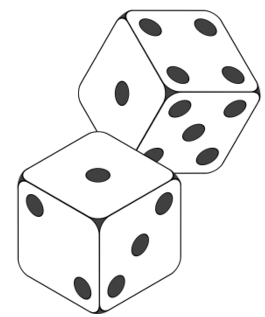


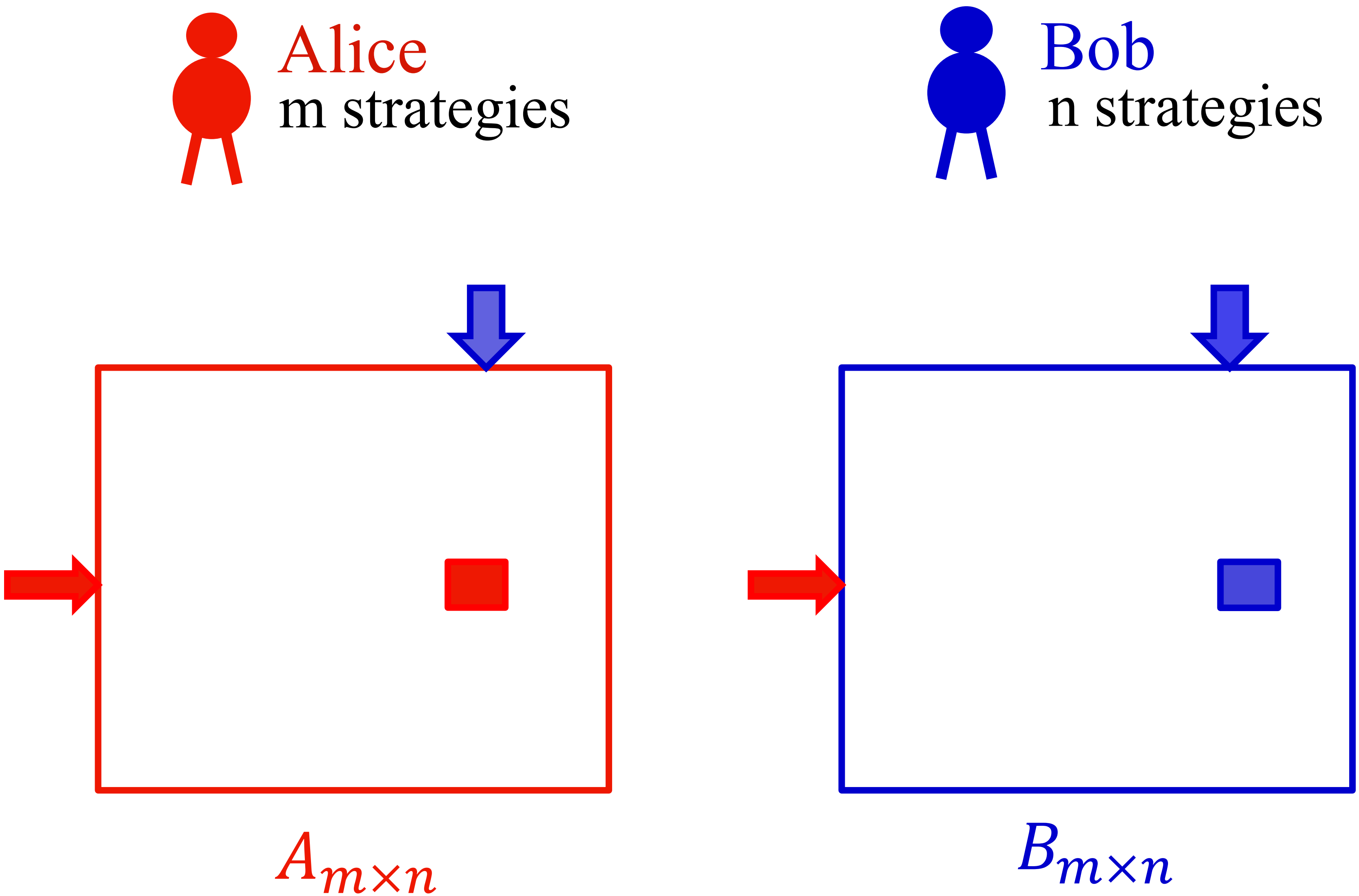
Randomize!

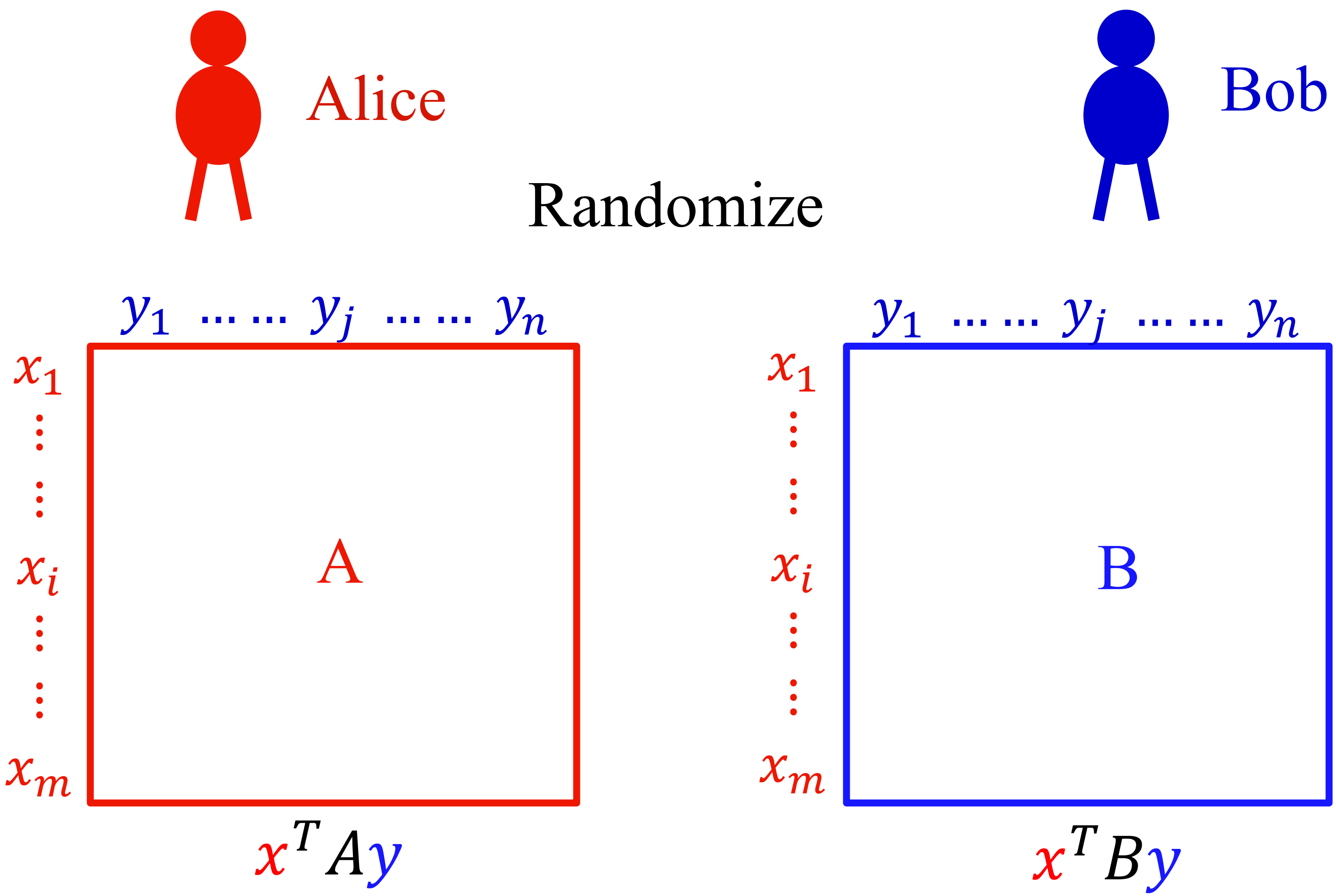
Nash (1950):

There exists a (stable) state where no player gains by unilateral deviation.

Nash equilibrium (NE)

Our focus: Two-player games





NE: No unilateral deviation is beneficial

$$x^T A y \geq z^T A y, \qquad \forall z \in \Delta_m$$

$$x^T B y \geq x^T B z, \qquad \forall z \in \Delta_n$$

Example

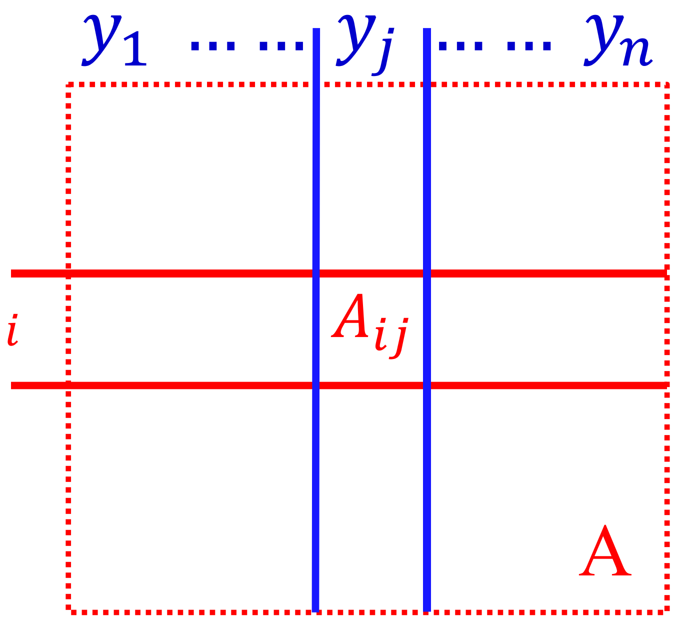
	R	P	S
R	0 0	-1 1	1 -1
P	1 -1	0 0	-1 1
S	-1 1	1 -1	0 0

2-Nash Characterization

2-Nash Characterization

Alice

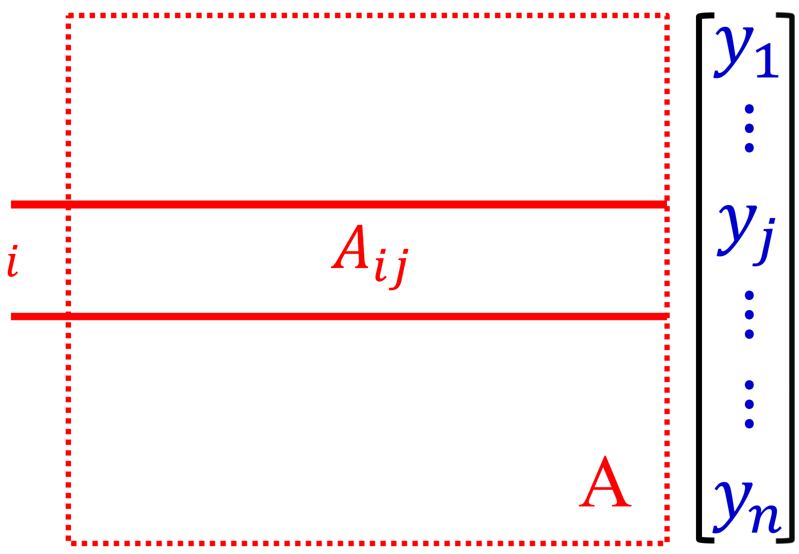
- For Row, i^{th} strategy gives



$$\longrightarrow \sum_j A_{ij} y_j$$

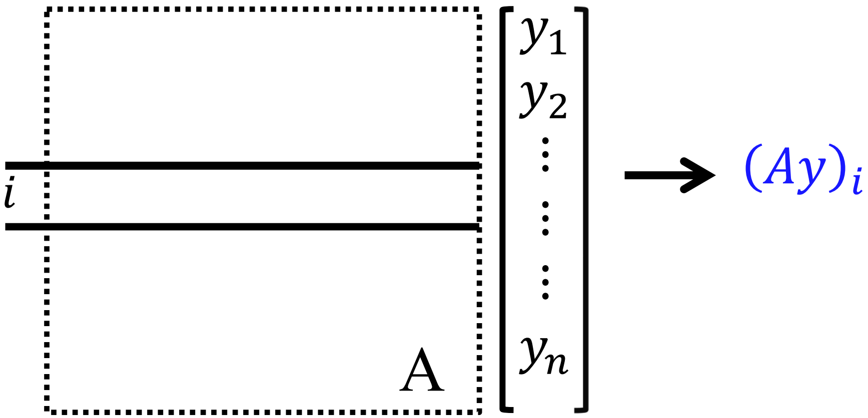
2-Nash Characterization

- For Row, i^{th} strategy gives



$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

- i^{th} strategy gives Alice



- Max possible payoff: $\max_i e_i A y$
- x achieves max payoff iff

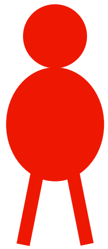
$$\forall i, \qquad x^T A y \geq (Ay)_i$$

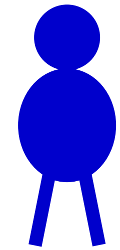
$$\equiv$$

$$\forall k, \qquad x_k > 0 \Rightarrow (Ay)_k = \max_i (Ay)_i$$

Complementarity

Polyhedra





max-payoff $\leq \pi_A$

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}}$$

max-payoff $\leq \pi_B$

$$Q \quad \boxed{\begin{array}{c} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}}$$

P

$$\forall i, (Ay)_i \le \pi_A$$
$$y \in \Delta_n$$

Q

$$\forall j, (x^T B)_j \le \pi_B$$
$$x \in \Delta_m$$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Sum of payoffs

At least the sum of max payoffs

$$\underbrace{x^T (A+B) y}_{\text{Sum of payoffs}} - \underbrace{(\pi_A + \pi_B)}_{\text{At least the sum of max payoffs}} \le 0$$

$$P \quad \begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{c} \forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m \end{array}$$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Sum of payoffs

At least the sum of max payoffs

$$\underbrace{x^T(A+B)y}_{\text{Sum of payoffs}} - \underbrace{(\pi_A + \pi_B)}_{\text{At least the sum of max payoffs}} \quad \mathbf{= 0}$$

$\Updownarrow$

Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

P $\boxed{\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}}$

Q $\boxed{\begin{array}{c} \forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m \end{array}}$

Claim. For $(y, \pi_A, x, \pi_B) \in P \times Q$, (i) $x^T(A + B)y - (\pi_A + \pi_B) \le 0$.
(ii) $x^T(A + B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n\end{array}}$

Q $\boxed{\begin{array}{c}\forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m\end{array}}$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

2-Nash

Sum of payoffs

At least the sum of max payoffs

$$\text{max: } \overbrace{x^T(A+B)y} - \overbrace{(\pi_A + \pi_B)} \quad \boxed{\mathbf{= 0}}$$

$$\text{s.t. } (y, \pi_A, x, \pi_B) \in P \times Q$$

Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

P $\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$

Q $\begin{array}{c} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

max: $x^T(A + B)y - (\pi_A + \pi_B)$

s.t. $(y, \pi_A, x, \pi_B) \in P \times Q$

P $\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}$

Q $\begin{array}{c} \forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m \end{array}$

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Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\text{max: } -(\pi_A + \pi_B)$$
$$\text{s.t. } (y, \pi_A, x, \pi_B) \in P \times Q$$

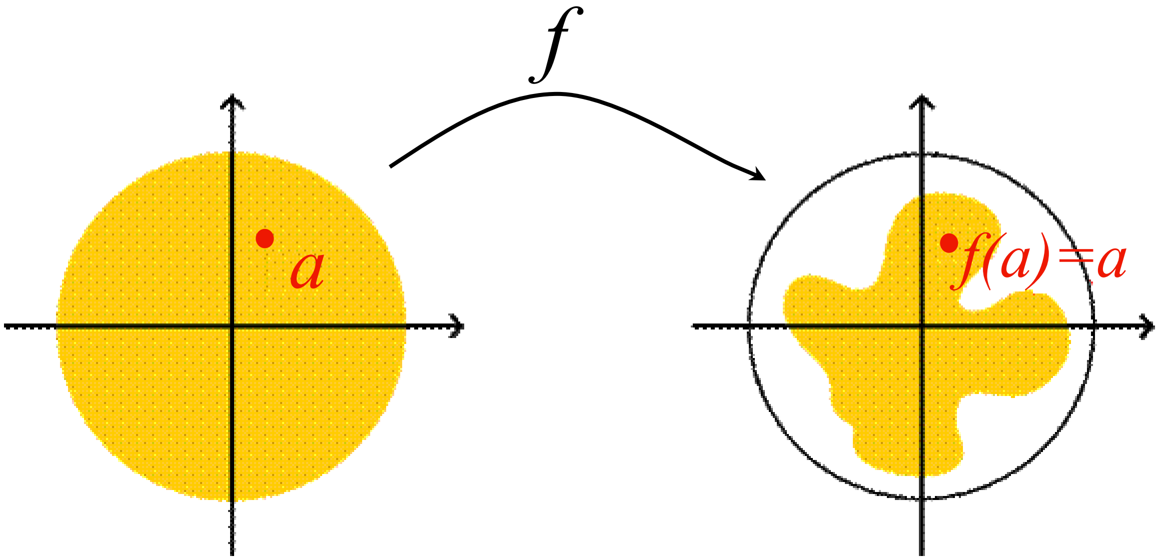
Theorem. [von Neumann'28] (max-min = min-max)

Game $(A, -A) \equiv$ wrt A, Alice is a maximizer and Bob minimizer

$\max_x \min_y x^T A y = \min_y \max_x x^T A y$ & the max-min is NE.

Computation in general?

NE existence via fixed-point theorem.



Computation? (in Econ)

- Special cases: Dantzig'51, Lemke-Howson'64, Elzen-Talman'88, Govindan-Wilson'03, …

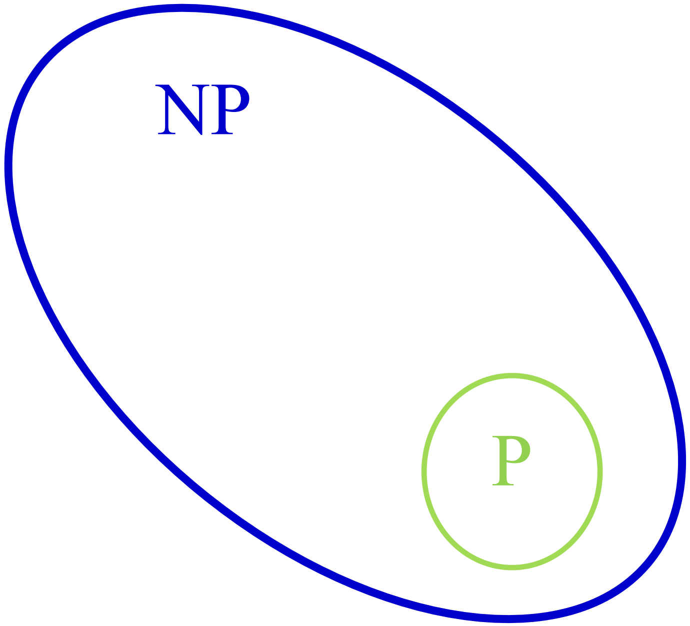
- Scarf'67: Approximate fixed-point.
  - Numerical instability
  - Not efficient!

- …

Computation? (in CS)

Not easy!

∃ solution?



What if solution always exists, like Nash Eq.?

Computation? (in CS)

Megiddo and Papadimitriou'91 :

Nash is NP-hard $\Rightarrow$ NP=Co-NP

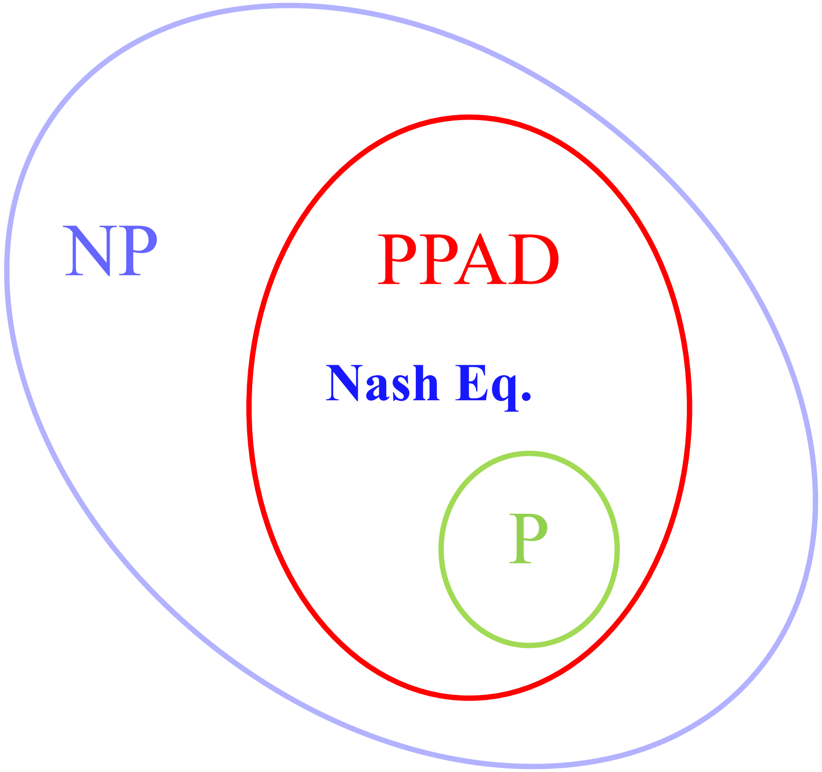
NP-hardness is ruled out!

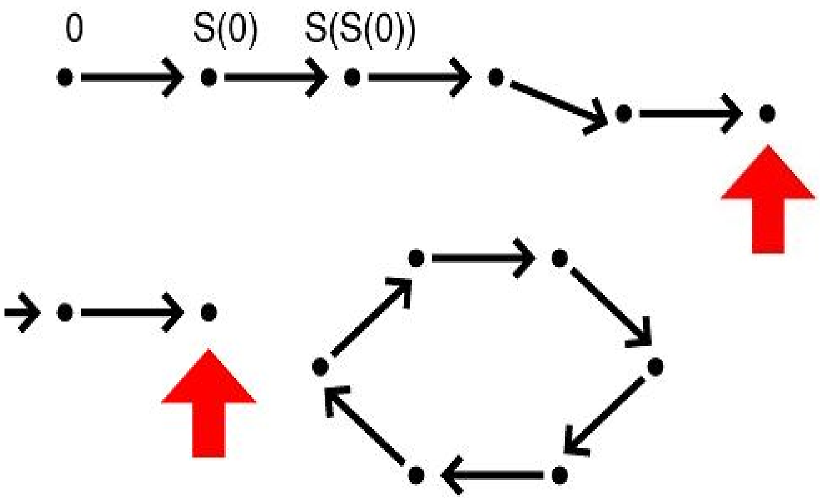
Complexity Classes

2-Nash is PPAD-complete!
[DGP'06, CDT'06]

Papadimitriou'94

PPAD Polynomial Parity Argument for Directed graph





Find an end

P $\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$

Q $\begin{array}{c} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\text{max: } -(\pi_A + \pi_B)$$
$$\text{s.t. } (y, \pi_A, x, \pi_B) \in P \times Q$$

Rank of a game: rank(A+B)

Zero-sum ≡ Rank-0 games

Rank 1 Game

$$A + B = u \cdot v^T$$

$u \in R^m$, $v \in R^n$

Bilinear

2-Nash

max: $x^T(A+B)y - (\pi_A + \pi_B)$

s.t. $(y, \pi_A, x, \pi_B) \in P \times Q$

Rank 1 Game [AGM.S'11]

$$A + B = u \cdot v^T$$

Product of two linear terms

2-Nash

$$\text{max:} \quad (x^T u)(v^T y) - (\pi_A + \pi_B)$$

$$\text{s.t.} \quad (y, \pi_A, x, \pi_B) \in P \times Q$$

Rank 1 QP is NP-hard in general

Rank 1 Game [AGM.S'11]

$$A + B = u \cdot v^T \longrightarrow (A, u, v)$$

2-Nash

$$\text{max:} \quad (x^T u)(v^T y) - (\pi_A + \pi_B)$$

$$\text{s.t.} \quad P \times Q$$

$$(x^T u)v^T - x^T A$$

Think Big!

Consider game space $S = (A, *, v)$

2-Nash

max: $(x^T *)(v^T y) - (\pi_A + \pi_B)$

s.t. $P \times Q$

$$(x^T *)v^T - x^T A$$

Think Big!

Consider game space $S = (A, *, v)$ All NE of S

Complementarity

Captures

LP(λ)

$$\max: \lambda(v^T y) - (\pi_A + \pi_B)$$

s.t. $P \times Q$

$$\lambda v^T - x^T A$$

Solutions of LP(λ) $\forall \lambda \in \mathbb{R}$

Claim. For any $\lambda \in \mathbb{R}$, optimal value of LP(λ) is zero.

max: $\lambda(v^T y) - (\pi_A + \pi_B)$

s.t. $P \times Q$

$\downarrow$

LP(λ) $\lambda v^T - x^T A$

λ

↓

LP(λ)

↓

(y, π_R, x, π_C)

Goal: NE of $(R, \boldsymbol{u}, v)$

⇓

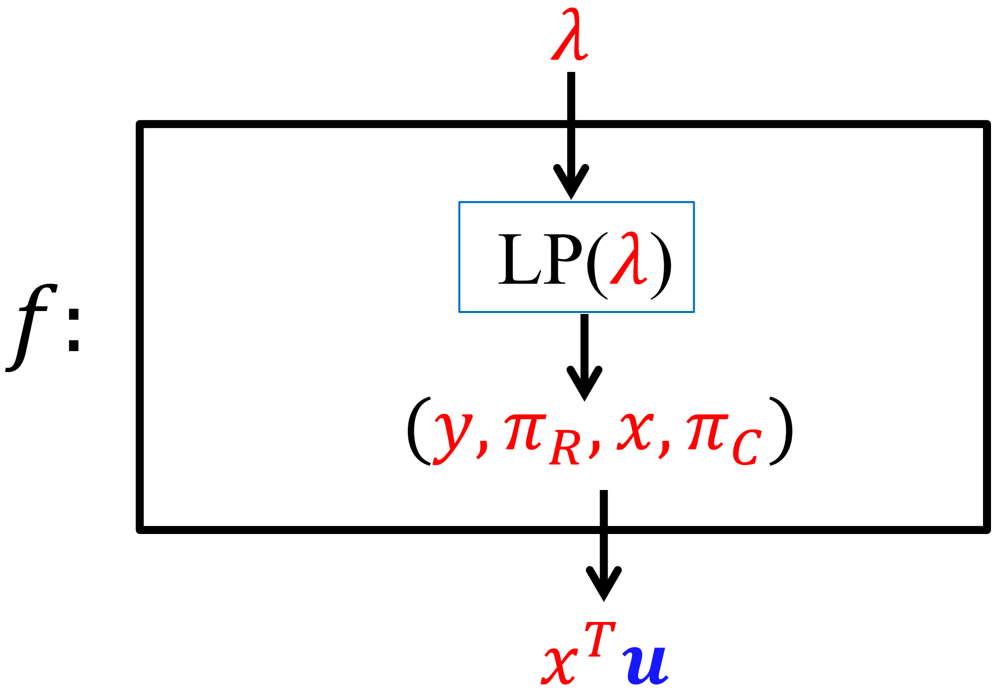
If $\boldsymbol{u}$ one of them then done!

(m-1)-dimensional space in S

Claim: $\forall \boldsymbol{c}$ s.t. $x^{\mathrm{T}}\boldsymbol{c} = \lambda$,

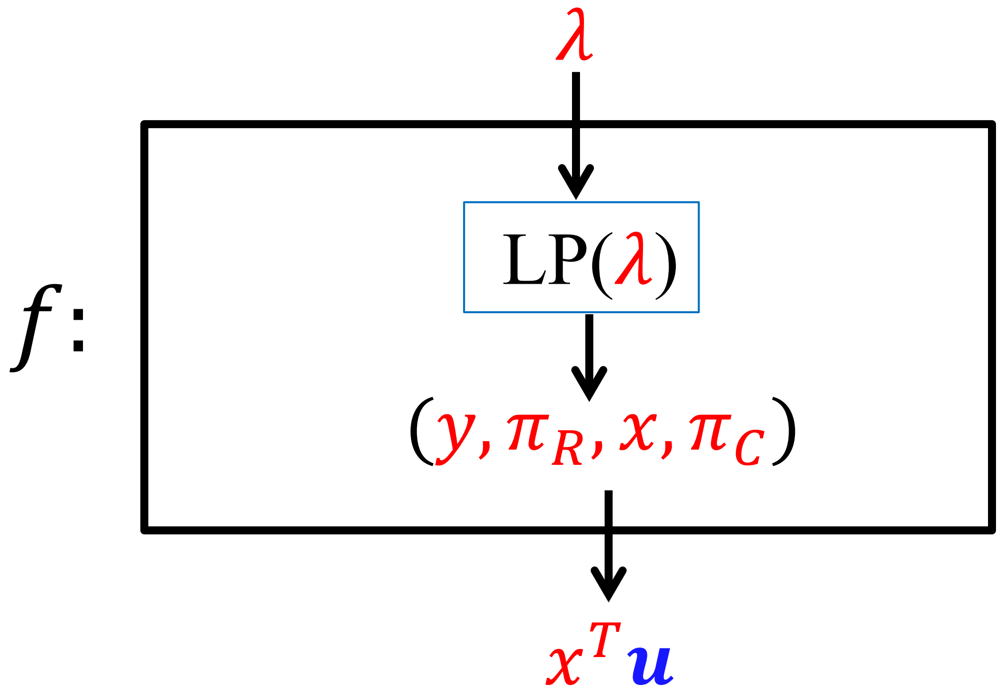
(x, y) is a NE of game $(R, \boldsymbol{c}, v)$

Goal: NE of game $(R, \boldsymbol{u}, v)$



If $x^T\boldsymbol{u} = \lambda$ then done!

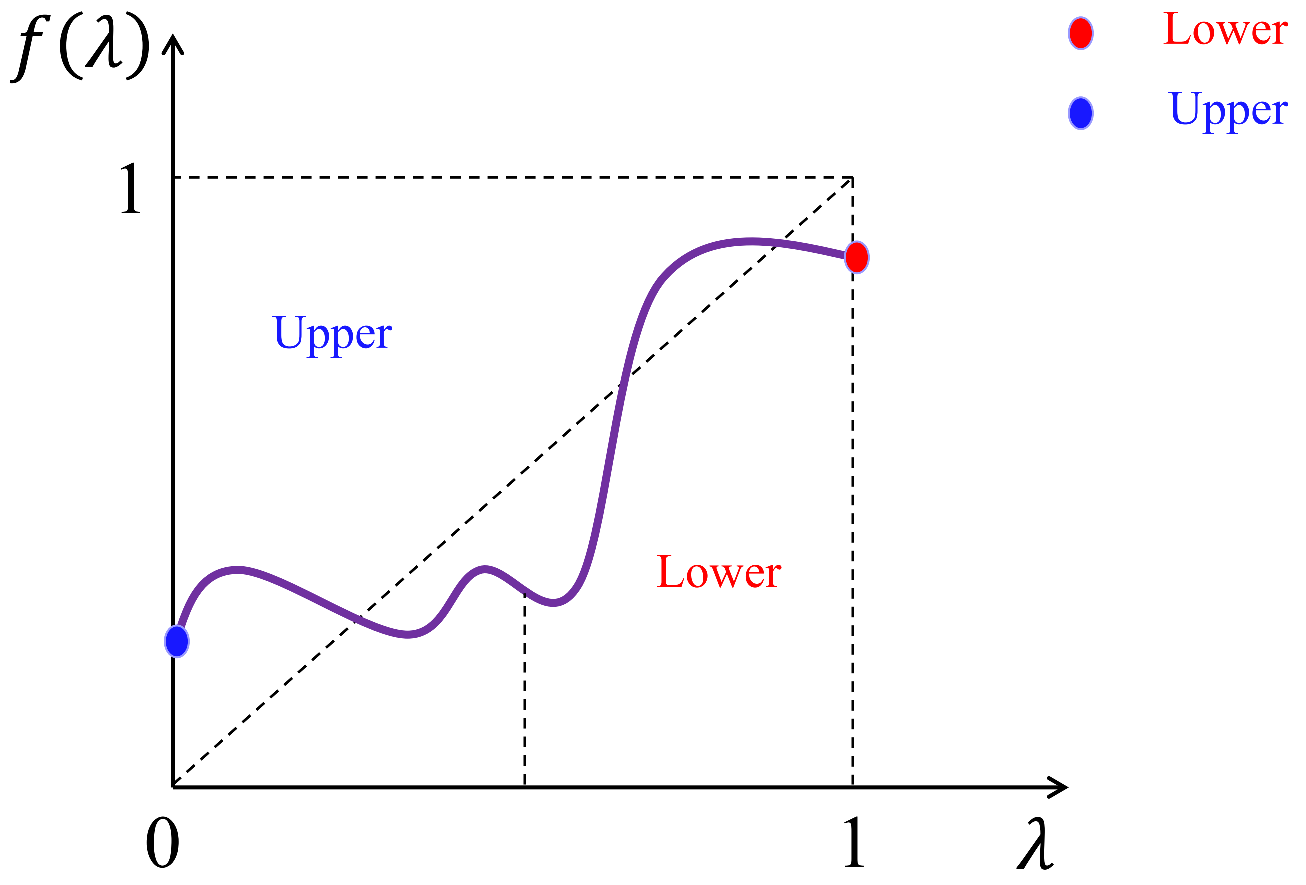
Goal: NE of game $(R, \boldsymbol{u}, v)$



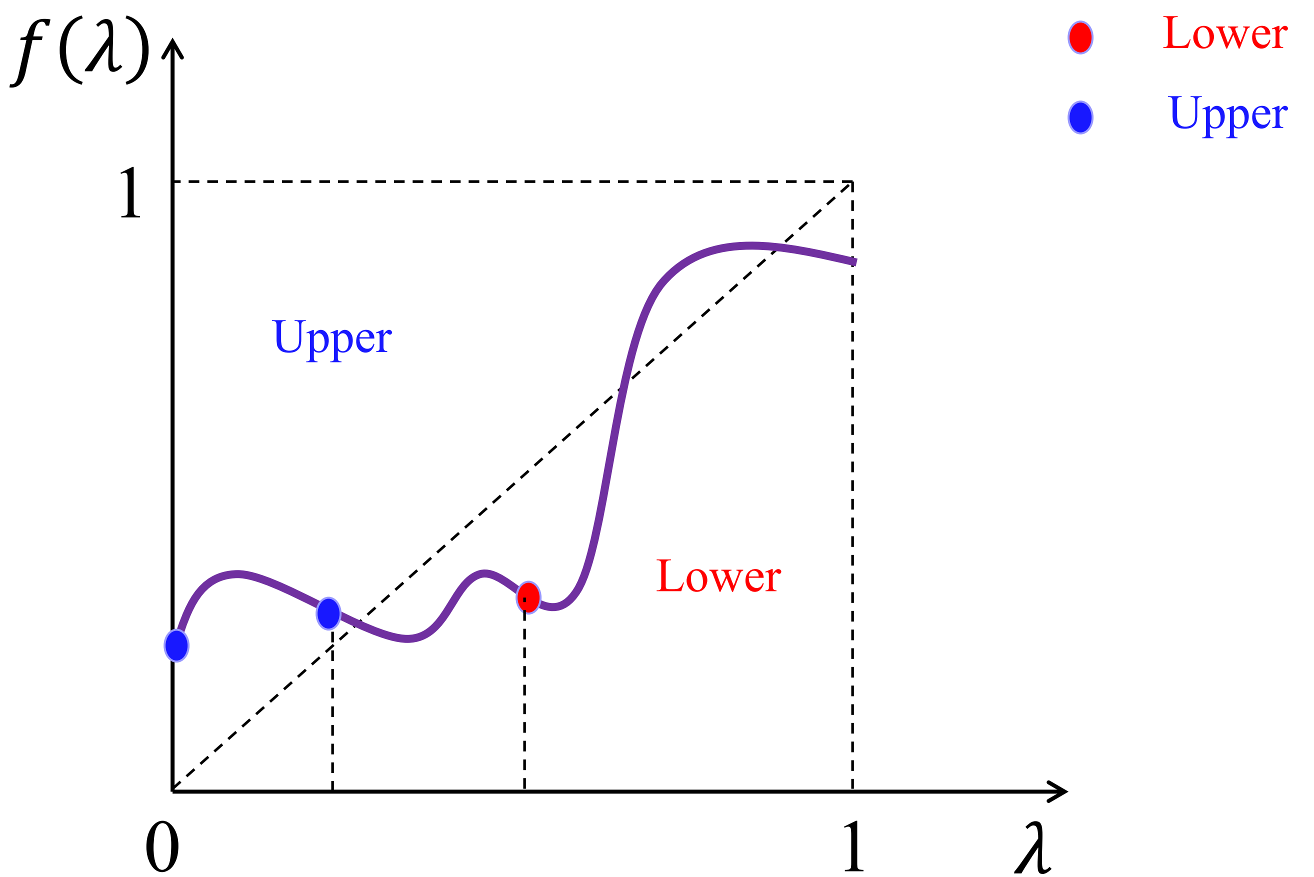
NE of game $(R, \boldsymbol{u}, v) \leftarrow$ Fixed points of f

If $x^T \boldsymbol{u} = \lambda$ then done!

1-D Fixed Point



1-D Fixed Point



And so on until the difference becomes small enough

What about rank-2 or more?

Rank-0 (zero-sum) games rank(A+B)=0 → Von Neumann (1928) → LP

Rank-1 games rank(A+B)=1 →

λ

$\underline{x} = \mathrm{LP}(\lambda)$

$\underline{c}^T \underline{x}$

Rank-2 games rank(A+B)=2 →

λ_1 λ_2

$\underline{x} = \mathrm{LP}(\lambda_1, \lambda_2)$

$\underline{c}^T \underline{x}$ $\underline{d}^T \underline{x}$

⋮

⋮

Rank-0 (zero-sum) games → (Von Neumann (1928)) → LP

Rank-1 games → 1-D Fixed Point

In P → [LP, 1-D Fixed Point]

Rank-2 games → 2-D Fixed Point

⋮

Rank-0 (zero-sum) games → LP

Von Neumann (1928)

Rank-1 games → 1-D Fixed Point

PPAD-hard in general

Rank-2 games → 2-D Fixed Point

⋮

⋮

Rank-0 (zero-sum) games → LP
Von Neumann (1928)

Rank-1 games → 1-D Fixed Point

PPAD-hard in general → 2-D Fixed Point

Rank-2 games ← ? 2-D Fixed Point

⋮

⋮

Rank-0 (zero-sum) games → Von Neumann (1928) → LP

Rank-1 games → 1-D Fixed Point

PPAD-hard in general

Rank-2 games ← [M.'14, COPY'16] ← 2-D Fixed Point

⋮

⋮