Lecture 7
Fair Division w/ Indivisible Items (Contd.)

CS 598RM

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Fairness Notions for Indivisible Items

- n agents, m indivisible items (like cell phone, painting, etc.)
- Each agent i has a valuation function over subset of items denoted by $v_i : 2^m \to \mathbb{R}$
- Goal: fair and efficient allocation

Fairness:
  Envy-free (EF)
  Proportionality (Prop)

Efficiency:
  Pareto optimal (PO)

Maximum Nash Welfare (MNW)

EF1	EFX
Prop1	MMS
Guarantees	

Allocation of Indivisible Items to Agents

- Set M of m indivisible items
- Set N of n agents
- Allocation $A = (A_1, \dots, A_n)$ is a partition of items to agents where each item is assigned to at most one agent

Objectives

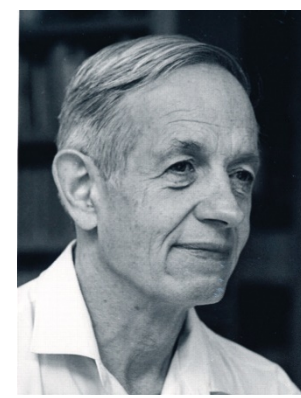
- Maximize the sum of valuations (**Utilitarian** Welfare): $SW(A) = \sum_i v_i(A_i)$

- Maximize the minimum of valuations (Max-Min-Fairness, **Egalitarian** Welfare):

$$SW(A) = \min_i v_i(A_i)$$

- Maximize the geometric mean of valuations (≈ **Efficiency** + **Fairness**, **Maximum Nash Welfare**):

$$\mathrm{NW}(A) = \left(\prod_{i \in A} v_i(A_i)\right)^{1/n}$$



Scale invariant

Maximum Nash Welfare (MNW)

- Maximum Nash welfare (MNW): An allocation A that maximizes the Nash welfare among all feasible allocations i.e.,

$$A^* = \arg\max_A \left(\prod_i v_i(A_i)\right)^{1/n}$$

Additive Valuations ($v_i(A_i) = \sum_{j \in A_i} v_{ij}$):

- Divisible Items: MNW ≡ CEEI ⇒ Envy-free + Prop + PO + …
- Indivisible Items: MNW ⇒ EF1 + PO + $\Omega(\frac{1}{\sqrt{n}})$-MMS [CKMPSW16]
  - Existence of EF1 + PO allocation

MNW: Generalizations

- Non-symmetric Agents (different entitlements/weights)
  - Weighted envy-free, weighted proportionality
  - MNW (weighted geometric mean)

- Beyond Additive Valuations

Additive $\subset$ GS $\subset$ Submodular $\subset$ XOS $\subset$ Subadditive

MNW: Generalizations

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The non-symmetric MNW Problem

- Non-symmetric MNW was proposed in [HS72, K77] and has been extensively studied and used in many applications
  - Agent i has a weight of w_i
- Allocation $A = (A_1, \dots, A_n)$ is partition of items to agents

$$\mathrm{NW}(A) = \left(\prod_i v_i(A_i)^{w_i} \right)^{1/\sum_i w_i}$$

weighted geometric mean of agents' valuations

- A^*: allocation maximizing the NW
- ρ-approximate MNW allocation A satisfies:

$$\rho.\mathrm{NW}(A) \geq \mathrm{NW}(A^*) = \mathrm{MNW}$$

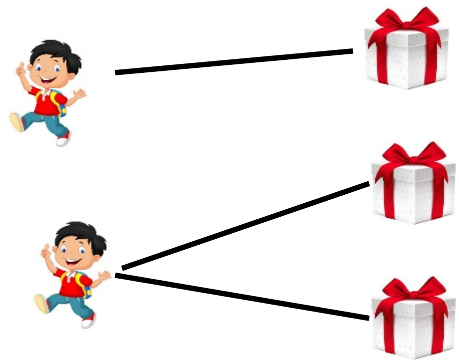
Example (additive)

w_i

A

1 [10, 10, 1]

1 [1, 2, 1]



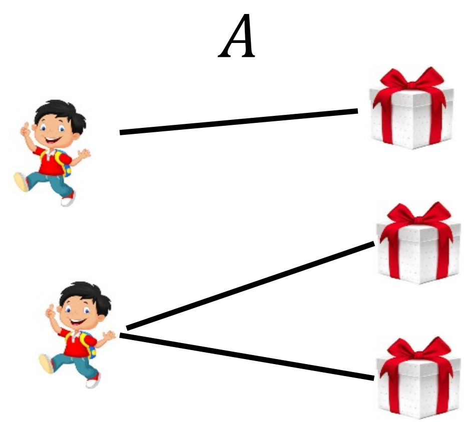
$$\text{MNW}=\text{NW}(A) = (10^1 \cdot 3^1)^{1/2}$$

Example (additive)

w_i

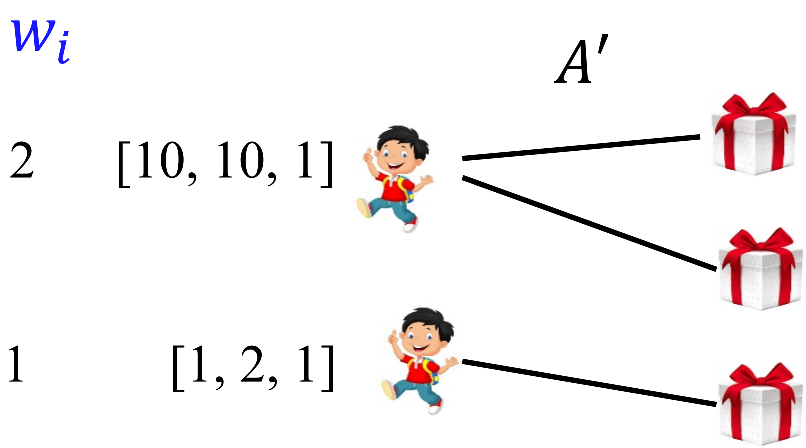
2 [10, 10, 1]

1 [1, 2, 1]



$$\mathrm{NW}(A) = (10^2 \cdot 3^1)^{1/3}$$

Example (additive)

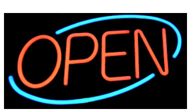


$$\mathrm{NW}(A) = (10^2 \cdot 3^1)^{1/3} < (20^2 \cdot 1^1)^{1/3} = \mathrm{NW}(A') = \mathrm{MNW}$$

MNW Approximations: Additive

	Lower bound	**Upper Bound**
Symmetric	1.069	1.45
Non-symmetric	1.069	$O(n)$

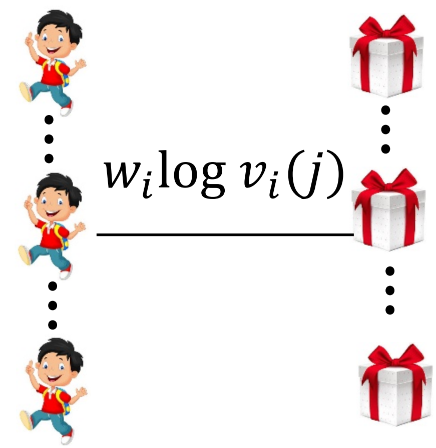
n: # of agents



Constant factor? sublinear?

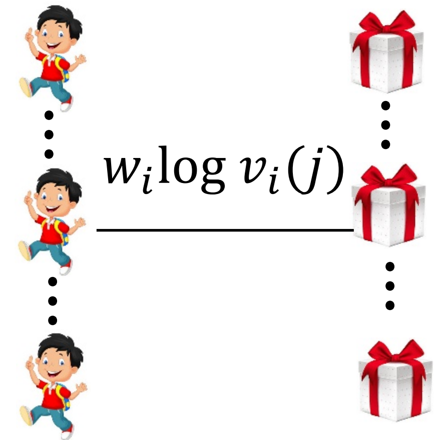
$m = n$: Matching

$$\text{NW}(A) = \left(\prod_i v_i(A_i)^{w_i}\right)^{1/\sum_i w_i}$$



$m = n$: Matching

$$\text{NW}(A) = \left(\prod_i v_i(A_i)^{w_i}\right)^{1/\sum_i w_i}$$

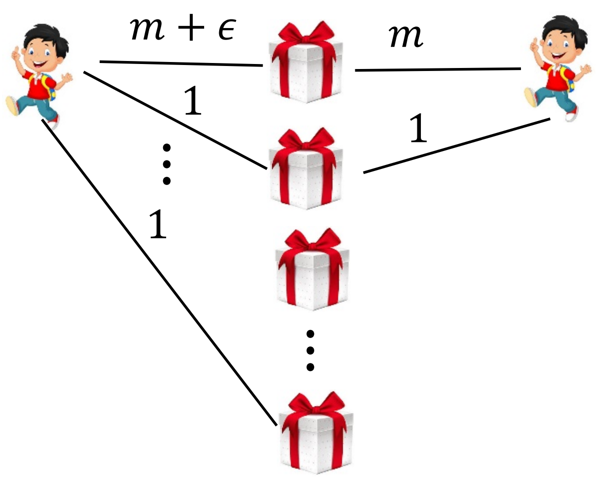


$$\text{MNW} = \max_A \ \text{NW}(A) \ \equiv \ \max_A \sum_i w_i \log v_i(A_i)$$

Claim: If $m = n$, then max-weight matching outputs MNW

$m > n$

- How good is max-weight matching?



$\mathrm{NW}(A^*) \simeq m$

$\mathrm{NW}(A) \simeq \sqrt{2m}$

- Issue: Allocation of high-value items!

Guarantee (per agent) at the optimum?

- $H_i = n$ highest-valued items of agent i. $u_i = v_i(M \setminus H_i)$
- g_i^*: highest-valued item in MNW allocation A_i^*

$$v_i(A_i^*) = v_i(A_i^* \cap H_i) + v_i\big(A_i^* \cap (M \setminus \mathrm{H}_i)\big)$$
$$\leq \; n v_i(g_i^*) + \; u_i \; = n(v_i(g_i^*) + \frac{u_i}{n})$$

- If we obtain an allocation A such that $v_i(A_i) \geq v_i(g_i^*) + \frac{u_i}{n}$, then A is $O(n)$-approximation!

Round Robin Procedure

Guarantee (per agent) ?

- $H_i = n$ highest-valued items of agent i. $u_i = v_i(M \setminus H_i)$

$O(n)$-MNW + EF1 [GKK20]

- $H_i = 2n$ highest-valued items for agent i. $H = \cup_i H_i$
- $u_i = v_i(M \setminus H_i)$
- Allocate as per max-weight matching from H with weights $w_i \log(v_i(g) + \frac{u_i}{n})$: y_i^* is allocated to i
- $A \leftarrow$ Allocate $M \setminus (\cup_i y_i^*)$ using round-robin procedure

Claim. $v_i(A_i) \geq v_i(y_i^*) + \frac{u_i}{n}$

- $H_i = 2n$ highest-valued items for agent i
- $u_i = v_i(M \setminus H_i)$
- Allocate one item to each agent using max-weight matching with weights $w_i \log(v_i(g) + \frac{u_i}{n})$: y_i^* is allocated to i
- $A \leftarrow$ Allocate remaining items using round-robin procedure

Claim. $v_i(A_i) \geq v_i(y_i^*) + \frac{u_i}{n}$

- g_i^*: highest-valued item in MNW allocation A_i^*
- $v_i(A_i^*) \leq 2n v_i(g_i^*) + u_i \leq 2n(v_i(g_i^*) + \frac{u_i}{n})$

$$\Rightarrow NW(A) \geq \left(\Pi_i \left(v_i(y_i^*) + \frac{u_i}{n}\right)^{w_i}\right)^{\frac{1}{\sum_i w_i}} \geq \left(\Pi_i \left(v_i(g_i^*) + \frac{u_i}{n}\right)^{w_i}\right)^{\frac{1}{\sum_i w_i}}$$

$$\geq \frac{1}{2n}\left(\Pi_i \left(v_i(A_i^*)\right)^{w_i}\right)^{\frac{1}{\sum_i w_i}}$$

Additive valuations are restrictive

100

Additive valuations are restrictive

100

100

Additive valuations are restrictive

100

+

$125 \neq 100 + 100$

100

Generalizations

- Non-symmetric Agents (different entitlements/weights)
  - Weighted envy-free, weighted proportionality
  - MNW (weighted geometric mean)

- Beyond Additive

Additive ⊂ GS ⊂ Submodular ⊂ XOS ⊂ Subadditive

non-negative monotone: $v(S) \leq v(T), \quad S \subseteq T$

Subadditive: $v(A \cup B) \leq v(A) + v(B), \quad \forall A, B$

Envy-free (EF) Allocation

Claim: An EF allocation A is $O(n)$-approximation

½-EFX:
Max-matching + Envy-cycle procedure

½-EFX Allocation

- ½-EFX allocation A: $v_i(A_i) \geq \frac{1}{2} v_i\left(A_j \setminus \mathrm{g}\right), \forall g \in A_j, \forall i, j$

Claim: If $|A_i| \geq 2, \forall i$, then A is $O(n)$-approximation

$O(n)$ Algorithm [CGM.20]

- H_i : n highest-valued items for agent i. $H = \cup_i H_i$
- $u_i = \frac{v_i(M \setminus \mathrm{H}_i)}{n}$
- Allocate as per max-weight matching from H with weights $w_i \log(v_i(g) + \frac{u_i}{n})$: y_i^* is allocated to i
- $Y = \cup_i y_i^*$
- $A \leftarrow$ Allocate $M \setminus \mathrm{Y}$ using ½-EFX algorithm

Claim: A is $O(n)$-MNW and ½-EFX allocation

$O(n)$ Algorithm

Claim: A is $O(n)$-MNW

Proof (sketch):

- g_i^*: highest-valued item in MNW allocation A_i^*
- $v_i(A_i^*) \leq \ n v_i(g_i^*) + v_i(M \setminus H_i) = n\left(v_i(g_i^*) + \frac{v_i(M \setminus H_i)}{n}\right)$
- $v_i(A_i) \geq v_i(y_i^*)$
- $v_i(A_i) \geq \frac{v_i(M \setminus Y)}{4n} \geq \frac{v_i(M \setminus H_i) - n v_i(y_i^*)}{4n}$

- $v_i(A_i) \geq \frac{1}{8}\left(v_i(y_i^*) + \frac{v_i(M \setminus H_i)}{n}\right) \geq \frac{1}{8}(v_i(g_i^*) + \frac{v_i(M \setminus H_i)}{n})$

MNW Approximations: Symmetric Agents

Additive ⊂ SC ⊂ OXS ⊂ GS / Budget additive ⊂ Submodular ⊂ XOS ⊂ Subadditive

Valuation	Lower bound	Upper Bound
Additive Budget additive Separable concave	1.069	1.45
OXS Gross-Substitutes	1.069	$O(1)^*$
Submodular XOS Subadditive	1.58	$O(n)$

*This is a very recent result [GHV20]

n: # of agents

MNW Approximations: Non-symmetric Agents

Additive ⊂ SC ⊂ OXS ⊂ GS / Budget additive ⊂ Submodular ⊂ XOS ⊂ Subadditive

Valuation	**Lower bound**	**Upper Bound**
Additive Budget additive Separable concave OXS Gross-Substitutes	1.069	$O(n)$
Submodular XOS Subadditive	1.58	$O(n)$

n: # of agents

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