Lecture 5: Fair Division w/ Indivisible Items

CS 598RM

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Fairness Notions for Indivisible Items

- n agents, m indivisible items (like cell phone, painting, etc.)
- Agent i has a valuation function $v_i : 2^m \to \mathbb{R}$ over subsets of items
- Goal: fair and efficient allocation

Fairness:
Envy-free (EF)
Proportionality (Prop)

Efficiency:
Pareto optimal (PO)

Maximum Nash Welfare (MNW)

EF1	EFX
Prop1	MMS
Guarantees	

Envy-Freeness up to One Item (EF1)

- An allocation $(A_1, \dots, A_n)$ is EF1 if for every agent i

$$v_i(A_i) \geq v_i(A_k \setminus g), \qquad \exists g \in A_k, \qquad \forall k$$

That is, agent i may envy agent k, but the envy can be eliminated if we remove a single item from $k's$ bundle

Envy-Freeness up to Any Item (EFX) [CKMPS14]

- An allocation $(A_1, \ldots, A_n)$ is EFX if for every agent i

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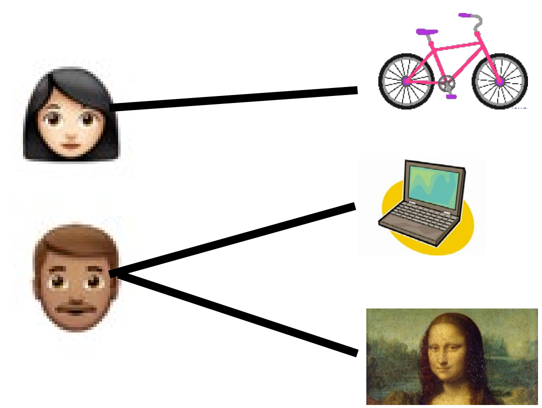
EF1 ?

EFX ?

[15, 10, 20]

[1, 20, 10]



EFX: Existence

- General Valuations [PR18]
  - Identical Valuations
  - $n = 2$

- Additive Valuations
  - $n = 3$ [CGM20]

Additive ($n > 3$), General ($n > 2$)

"Fair division's biggest problem" [P20]

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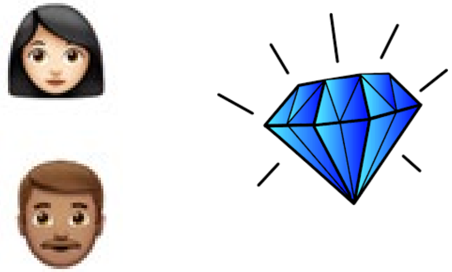
Proportionality up to One Item (Prop1)

- A set N of n agents, a set M of m indivisible items

- Proportionality (Prop): Allocation $A = (A_1, \ldots, A_n)$ is proportional if each agent gets at least $1/n$ share of all items:

$$v_i(A_i) \geq \frac{v_i(M)}{n}, \qquad \forall i \in N$$



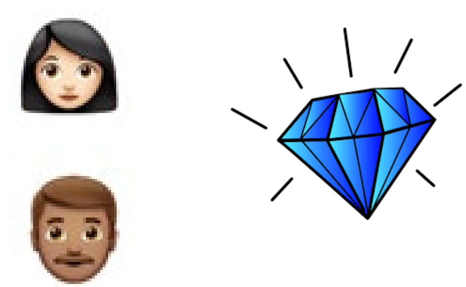
Proportionality up to One Item (Prop1)

- A set N of n agents, a set M of m indivisible items
- Prop: $A = (A_1, \dots, A_n)$ is proportional if each agent gets at least $1/n$ share of all items:

$$v_i(A_i) \geq \frac{1}{n} v_i(M), \qquad \forall i \in N$$

- Prop1: A is proportional up to one item if each agent gets at least $1/n$ share of all items after adding one more item from outside:

$$v_i(A_i \cup \{g\}) \geq \frac{1}{n} v_i(M), \qquad \exists g \in M \setminus A_i, \forall i \in N$$



Prop1

- EF1 implies Prop1 for subadditive valuations 

  $\Longrightarrow$ Envy-cycle procedure outputs a Prop1 allocation

- Additive Valuations
  - EF1 + PO allocation exists but no polynomial-time algorithm is known!
  - Prop1 + PO?

Prop1 + PO [BK19]

- (p, x): CEEI
- x is envy-free ⇒ proportional
- we can assume that support of x is a forest (set of trees)
- In each tree:
  - Make some agent the root
  - Assign each item to its parent agent

Theorem: The output of the above algorithm is Prop1 + PO

Prop1 + PO [BK19]

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Cut-and-choose?

Maximin Share (MMS) [B11]

Cut-and-choose.

- Suppose we allow agent i to propose a partition of items into n bundles with the condition that i will choose at the end
- Clearly, i partitions items in a way that maximizes the value of her least preferred bundle
- μ_i := Maximum value of $i's$ least preferred bundle

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- Clearly, i partitions items in a way that maximizes the value of her least preferred bundle
- $\mu_i :=$ Maximum value of $i's$ least preferred bundle

- $\Pi \coloneqq$ Set of all partitions of items into n bundles
- $\mu_i \coloneqq \max_{A \in \Pi} \min_{A_k \in A} v_i(A_k)$

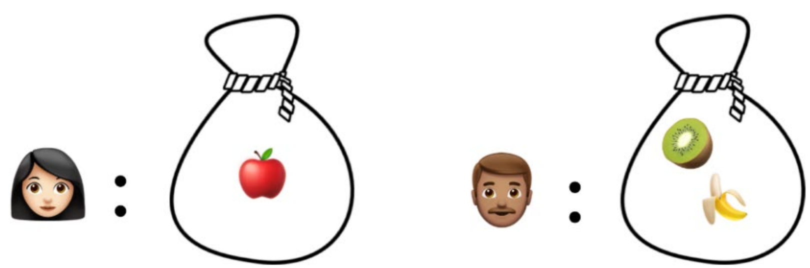
- MMS Allocation: A is called MMS if $v_i(A_i) \geq \mu_i, \forall i$
- Additive valuations: $v_i(A_i) = \sum_{j \in A_i} v_{ij}$

MMS value/partition/allocation

Agent\Items	🍎	🍌	🥝
👩	3	1	2
👨	4	4	5

👩	🍎	🍌🥝
Value	3	3
MMS Value	3	

👨	🍎🍌	🥝
Value	8	5
MMS Value	5	



MMS value/partition/allocation

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👩	3	1	2
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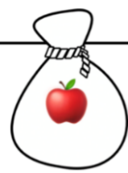
👩		
Value	3	3
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👨		
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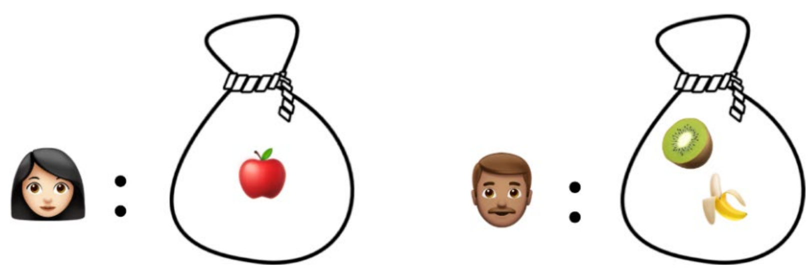

Finding MMS value is NP-hard!

What is Known?

- PTAS for finding MMS value [W97]

Existence (MMS allocation)?

- $n = 2$: yes 

  $\implies$ A PTAS to find $(1-\epsilon)$-MMS allocation for any $\epsilon > 0$
- $n \geq 3$: NO [PW14]

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- $n \geq 3$: NO [PW14]

- α-MMS allocation: $v_i(A_i) \geq \alpha . \mu_i$
  - 2/3-MMS exists [PW14, AMNS17, BK17, KPW18, GMT18]
  - 3/4-MMS exists [GHSSY18]
  - $(3/4 + 1/(12n))$-MMS exists [GT20]

Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \leq 1$

Properties

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- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

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	🍎	🍌	🍐	🍍	🥥
👩	3	1	2	5	4
👨	4	4	5	3	2

→

	1	**2**	**3**	**4**	**5**
👩	5	4	3	2	1
👨	5	4	4	3	2

Challenge

- Allocation of high-value items!
- If for all $i \in N$
  - $v_i(M) = n \Rightarrow \mu_i \leq 1$
  - $v_{ij} \leq \epsilon, \forall i, j$

Challenge

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Bag Filling Algorithm:

Repeat until every agent is assigned a bag

- Start with an empty bag B
- Keep adding items to B until some agent i values it $\geq (1 - \epsilon)$
- Assign B to i and remove them

Challenge

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Bag Filling Algorithm:

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Every agent gets at least $(1 - \epsilon)$!

Warm Up: 1/2-MMS Allocation

- Assume that μ_i is known for all i
  - Scale valuations so that $\mu_i = 1 \Rightarrow v_i(M) \geq n$
- If all $v_{ij} \leq 1/2$ then ?

Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \le 1$
- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \ge v_{i2} \ge \cdots v_{im}, \forall i \in N$
- Valid Reduction (α-MMS): If there exists $S \subseteq M$ and $i^* \in N$
  - $v_{i^*}(S) \ge \alpha.\mu_{i^*}^n(M)$
  - $\mu_i^{n-1}(M \setminus S) \ge \mu_i^n(M), \forall i \ne i^*$

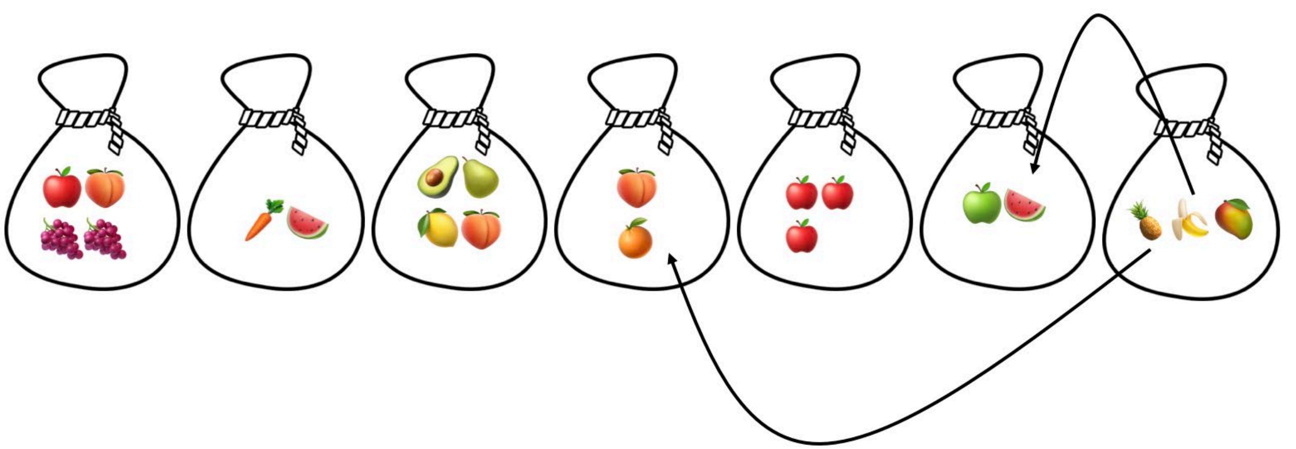
  $\Rightarrow$ We can reduce the instance size!

1/2-MMS Allocation

- Assume that μ_i is known for all i
  - Scale valuations so that $\mu_i = 1 \Rightarrow v_i(M) \geq n$

Step 1: Valid Reductions
- If $v_{i1} \geq 1/2$ then assign item 1 to i

Step 2: Bag Filling



1/2-MMS Allocation

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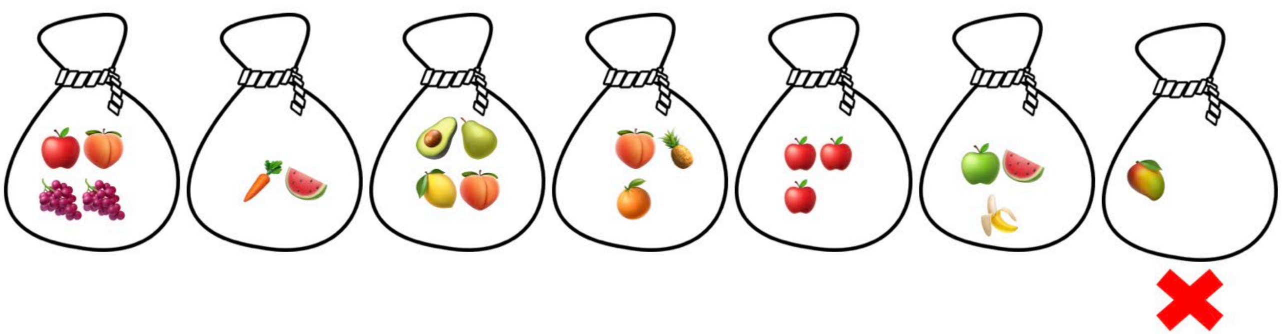


1/2-MMS Allocation

- μ_i is not known

Step 0: Normalized Valuations: $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$

Step 1: Valid Reductions

- If $v_{i1} \geq 1/2$ then assign item 1 to i
- After every valid reduction, normalize valuations

Step 2: Bag Filling

2/3-MMS Allocation [GMT19]

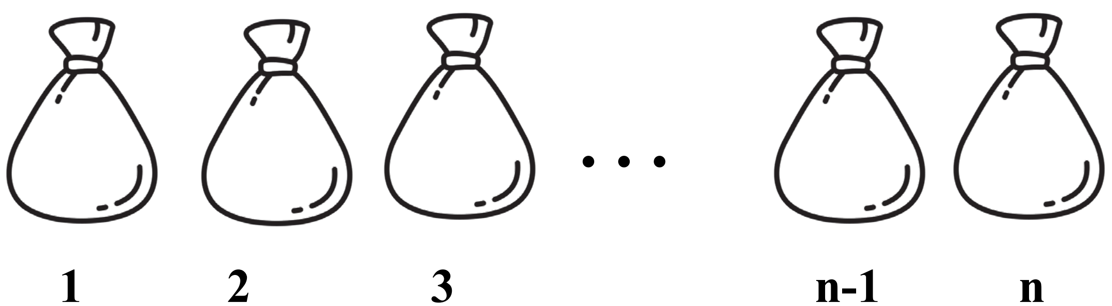
- Assume that μ_i is known for all i
  - Scale valuations so that $\mu_i = 1 \Rightarrow v_i(M) \geq n$
- If all $v_{ij} \leq 1/3$ then ?

Step 1: Valid Reductions

- If $v_{i1} \geq 2/3$ then assign item 1 to i
- If $v_{in} + v_{i(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i

Step 2: Generalized Bag Filling

- Initialize n bags $\{B_1, \ldots B_n\}$ with $B_k = \{k\}, \forall k$



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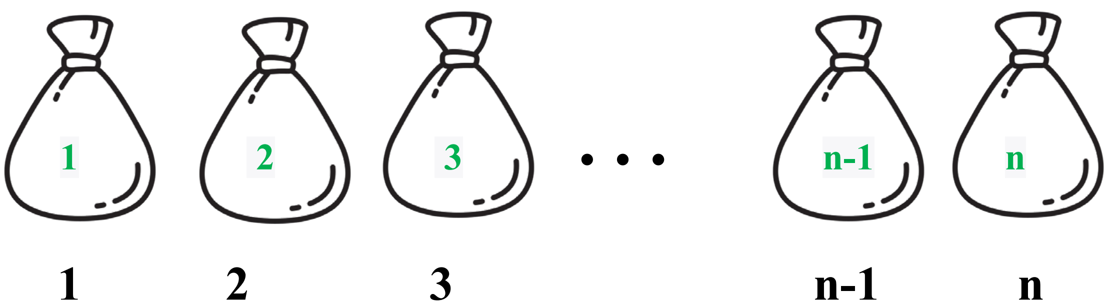

2/3-MMS Allocation [GMT19]

- μ_i is not known

Step 0: Normalized Valuations: $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$

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Summary

Covered

- Additive Valuations:
  - Prop1 + PO (polynomial-time algorithm)
  - 2/3-MMS allocation (polynomial-time algorithm)

Not Covered

- $\left(\frac{3}{4}+\right)$-MMS allocation [GT20]
- More general valuations
  - MMS [GHSSY18]
- Groupwise-MMS [BBKN18]
- Chores: 11/9-MMS [HL19]

Major Open Questions (additive)

- c-MMS + PO: polynomial-time algorithm for a constant $c > 0$
- Existence of 4/5-MMS allocation? For 5 agents?

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