Lecture 4: Fair Division w/ Indivisible Items

CS 598RM

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(Recall) Fisher's Model

- Set A of n agents. Set G of m divisible goods.
- Each agent i has
  - budget of B_i dollars
  - valuation function $v_i: R_+^m \to R_+$ over bundles of goods.

  Linear: for bundle $x_i = (x_{i1}, \dots, x_{im})$, $v_i(x_i) = \sum_{j \in G} v_{ij} x_{ij}$
- Supply of every good is one.

(Recall) Competitive Equilibrium

Pirces $p = (p_1, \dots, p_m)$ and allocation $X = (x_1, \dots, x_n)$

- Optimal bundle: Agent i demands
$$x_i \in \underset{x \in R_m^+ :\, p \cdot x \le B_i}{\operatorname{argmax}} \; v_i(x)$$

- Market clears: For each good j,
$$\text{demand} = \text{supply}$$

Fairness and efficiency guarantees:
Pareto optimal (PO)
Weighted Envy-free
Weighted Proportional
Maximizes W. NW.

[DPSV'08] Flow-based Algorithm

Efficient Flow-based Algorithms

- Polynomial running-time
  - Compute *balanced-flow:* minimizing l_2 norm of agents' surplus [DPSV'08]
- Strongly polynomial: Flow + scaling [Orlin'10]

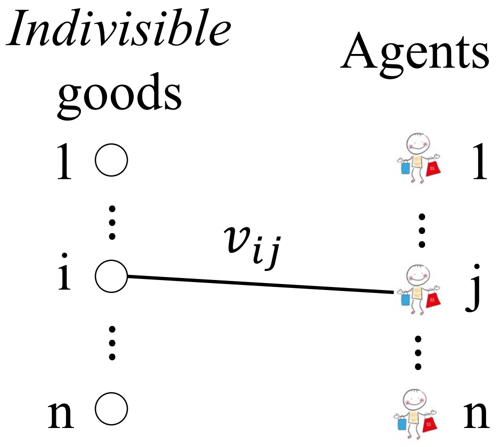
Exchange model (barter):

- Polynomial time [DM'16, DGM'17, CM'18]
- Strongly polynomial for exchange
  - Flow + scaling + approximate LP [GV'19]

Hylland-Zeckhauser

(an extension)

Motivation: Matching



Goal: Design a method to match goods to agents so that

- The outcome is **Pareto-optimal** and **envy-free**
- **Strategy-proof**: Agents have no incentive to lie about their $v_{ij}s$.

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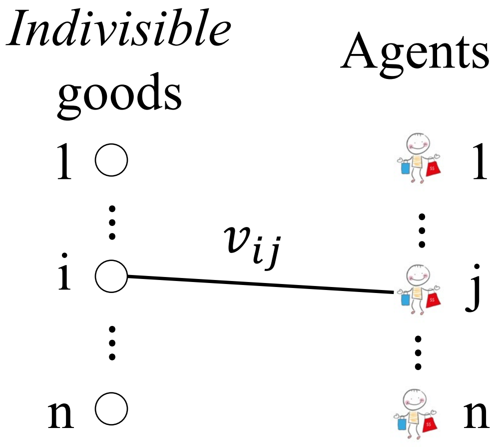

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Hylland-Zeckhauser'79: Compute CEEI where every agent wants total amount of at most one unit.

But the outcome is a fractional allocation!

Think of it as probabilities/time-shares/… [HZ'79, BM'04]

HZ Equilibrium

Given:

- Agents $A = \{1, \ldots, n\}$, indivisible goods $G = \{1, \ldots, n\}$
- v_{ij}: value of agent i for good j.
  - If i gets j w/ prob. x_{ij}, then the expected value is: $\sum_{j \in G} v_{ij} x_{ij}$

Want: prices $p = (p_1, \ldots, p_n)$, allocation $X = (x_1, \ldots, x_n)$

- Each good j is allocated: $\sum_{i \in A} x_{ij} = 1$
- Each agent i gets an optimal bundle subject to
  - $1 budget, and **unit allocation**.

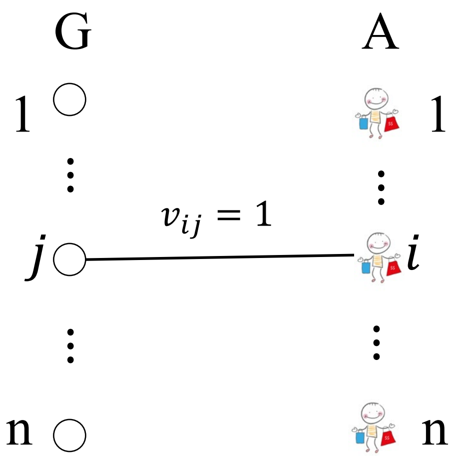
$$x_i \in \underset{x \in R_+^m}{\operatorname{argmax}} \left\{ \sum_j v_{ij} x_j \;\middle|\; \sum_j \boldsymbol{x_j} = \boldsymbol{1}, \sum_j p_j x_j \leq 1 \right\}$$

Exists. Pareto optimal, Strategy proof in large markets.

VY'20 Algorithm

$(v_{ij} \in \{0,1\})$

Want: (p, X)
All goods are sold.
Each agent i gets
$x_i \in \underset{x: \sum_j x_j = 1, \sum_j p_j x_j \le 1}{\text{argmax}} \sum_{j \in G} v_{ij} x_j$



At equilibrium, an agent's utility is at most 1.

Perfect matching ⇒ An equilibrium is,

- Allocation on the matching edges
- Zero prices

VY'20 Algorithm
($v_{ij} \in \{0,1\}$)

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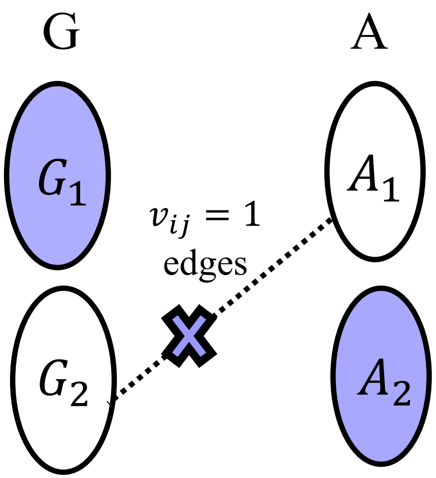

No perfect matching

- Min vertex cover: $(G_1 \cup A_2)$
  - No $A_1 - G_2$ edge

VY'20 Algorithm

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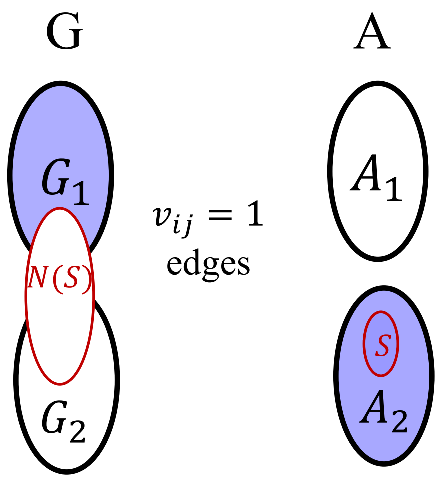

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No perfect matching

- Min vertex cover: $(G_1 \cup A_2)$
  - No $A_1 - G_2$ edge
  - For each $S \subseteq A_2$, $|N(S) \cap G_2| \geq |S|$
    - Else get smaller VC by replacing S with $N(S) \cap G_2$

⇓

Subgraph (G_2, A_2) satisfies hall's condition for A_2.

⇐

Max matching in (G_2, A_2) matches all of A_2.

VY'20 Algorithm

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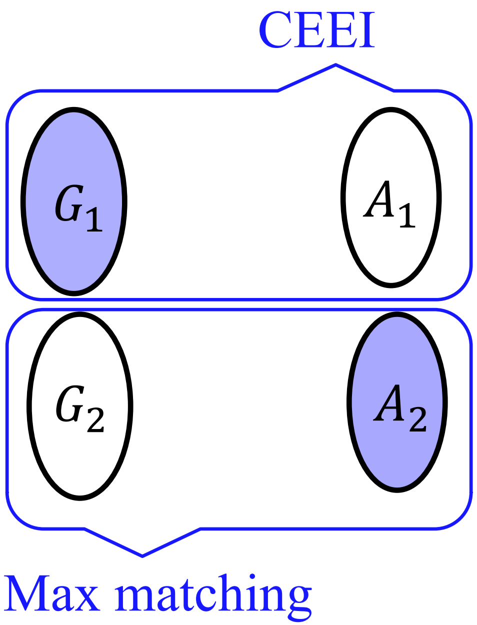

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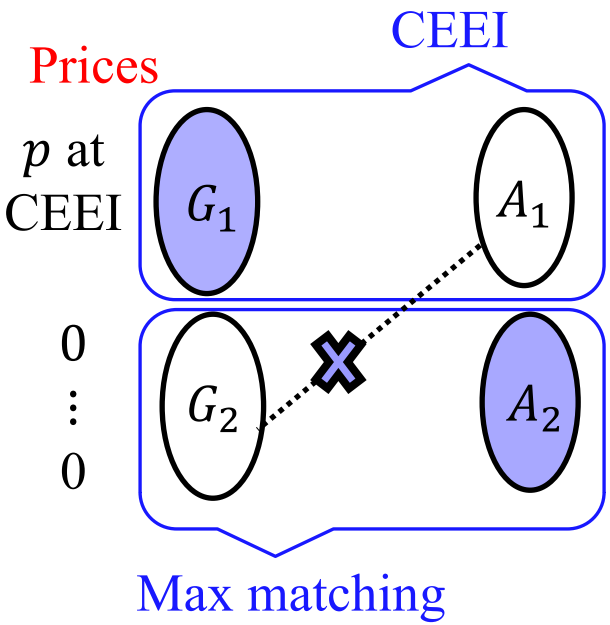

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No perfect matching

- Min vertex cover: $(G_1 \cup A_2)$
- Eq. Prices: CEEI prices for G_1, and
  0 prices for G_2
- Eq. Allocation
  - $i \in A_2$ gets her matched good
  - $i \in A_1$ gets CEEI allocation + unmatched goods from G_2

Running-time:
Strongly polynomial

VY'20 Algorithm

bi-values: $v_{ij} \in \{a_i, b_i\}, 0 \leq a_i < b_i$

Reduces to $v_{ij} \in \{0,1\}$

Exercise.

Open Questions

HZ Equilibrium

Computation for the general case.

Is it hard? OR is it (approximation) polynomial-time?

- Efficient algorithm when #goods or #agents is a constant [DK'08, AKT'17]
  - Cell-decomposition and enumeration

What about chores?

- CEEI exists but may form a non-convex set [BMSY'17]

- Efficient Computation?
  - **Open: Fisher as well as for CEEI**
  - For constantly many agents (or chores) [BS'19, GM'20]
  - *Fast* path-following algorithm [CGMM.'20]

- Hardness result for an exchange model [CGMM.'20]

Indivisible Items

UCLA Kidney Exchange Program

Indivisible Items

- n agents, m indivisible items (like cell phone, painting, etc.)
- Agent i has a valuation function $v_i : 2^m \rightarrow \mathbb{R}$ over subsets of items
- Goal: fair and efficient allocation

Fairness:

Envy-free (EF)

Proportionality (Prop)

Efficiency:

Pareto optimal (PO)

Maximum Nash Welfare (MNW)

Fairness Notions for Indivisible Items

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EF1	EFX
Prop1	MMS
Guarantees	

Envy-Freeness up to One Item (EF1) [B11]

- An allocation $(A_1, \ldots, A_n)$ is EF1 if for every agent i

$$v_i(A_i) \geq v_i(A_k \setminus g), \ \ \exists g \in A_k, \ \ \forall k$$

That is, agent i may envy agent k, but the envy can be eliminated if we remove a single item from $k's$ bundle

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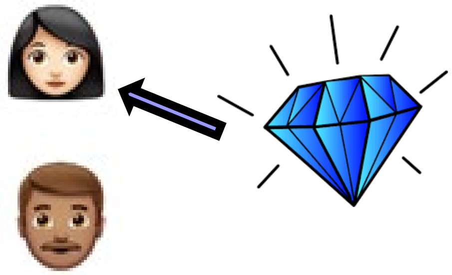
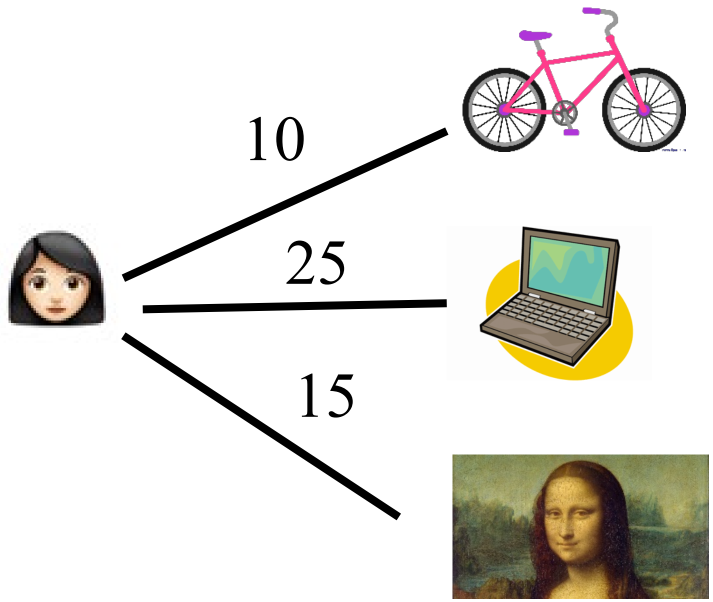
- Existence?



Additive Valuations: $v_i(S) = \sum_{j \in S} v_{ij}$



Round Robin Algorithm (Additive)

- Fix an ordering of agents arbitrarily
- While there is an item unallocated
  - i: next agent in the round robin order
  - Allocate i her most valuable item among the unallocated ones

Claim: The final allocation is EF1

Observe that intermediate (partial) allocation is also EF1

Envy-Cycle Procedure (General) [LMMS04]

- General Monotonic Valuations: $v_i(S) \leq v_i(T),\ \forall S \subseteq T \subseteq M$
  (M: Set of all items)

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- Envy-graph of a partial allocation $(A_1, \ldots, A_n)$ where $\cup_i A_i \subseteq M$
  - Vertices = Agents
  - Directed edge (i, i') if i **envies** i' (i.e., $v_i(A_i) < v_i(A_{i'})$)

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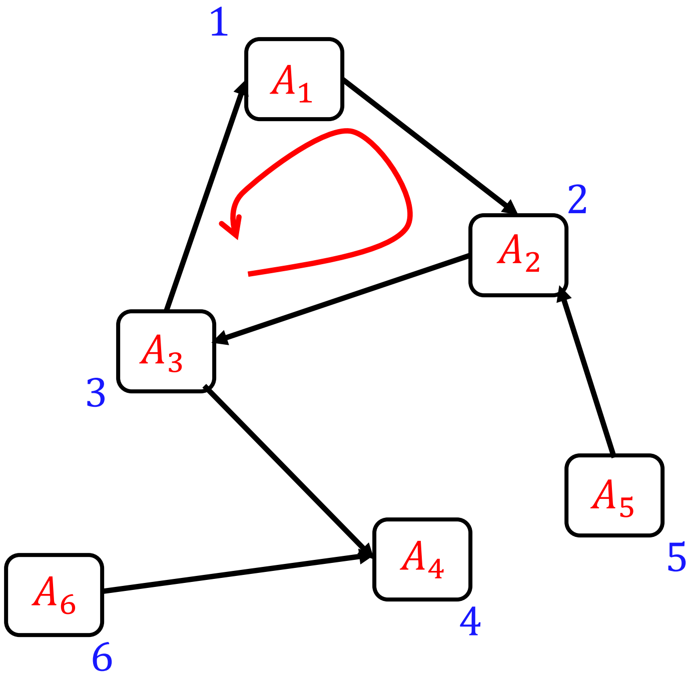
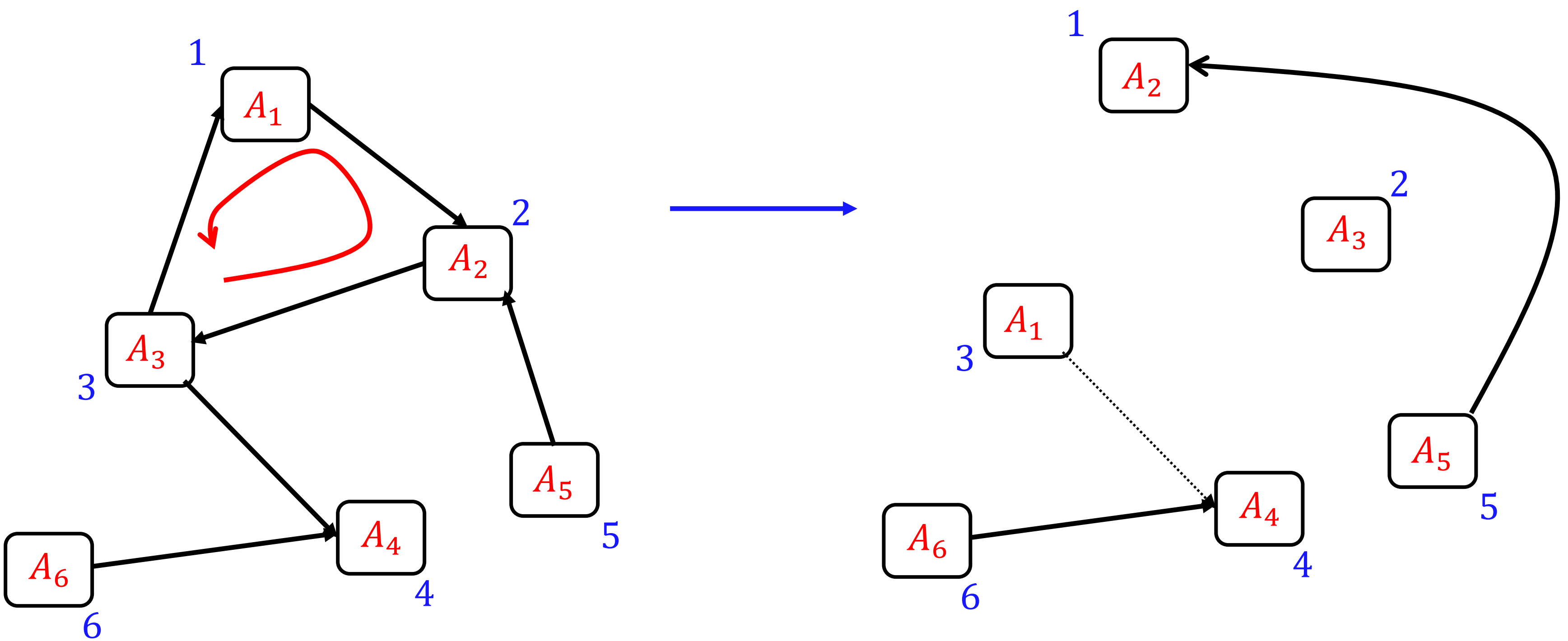
- Suppose we have a partial EF1 allocation
- Then, we can assign one unallocated item j to a source i (in-degree 0 agent) and the resulting allocation is still EF1!
  - No agent envies i if we remove j

- If there is no source in envy-graph, then
  - there must be cycles
  - How to eliminate them?



- If there is no source in envy-graph, then
  - there must be cycles
  - keep eliminating them by exchanging bundles along each cycle
- Terminate?



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  - Valuation of each agent?
  - The bundles remain the same – We are only changing their owners!

Envy-Cycle Procedure [LMMS04]

$A \leftarrow (\emptyset, \dots, \emptyset)$

$R \leftarrow M$ // unallocated items

While $R \neq \emptyset$

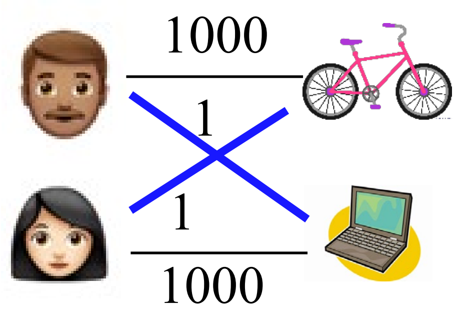
- If envy-graph has no source, then there must be cycles
- Keep removing cycles by exchanging bundles until there is a source
- Pick a source, say i, and allocate one item g from R to i

  $(A_i \leftarrow A_i \cup g;\ R \leftarrow R \setminus g)$

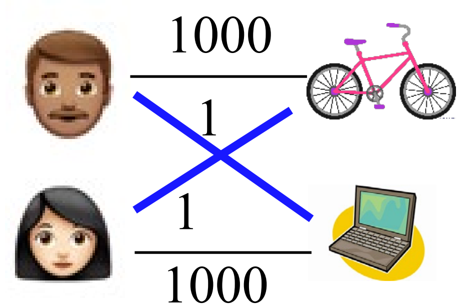
Output A

- Running Time? 

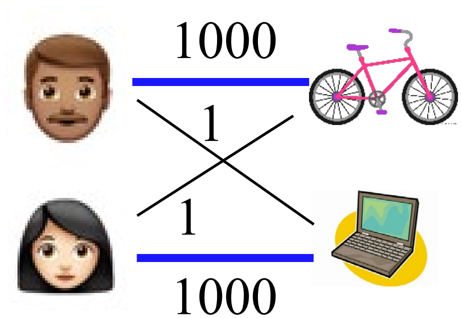
How Good is an EF1 Allocation?



How Good is an EF1 Allocation?



- Certainly not desirable!



- Issue: Many EF1 allocations!
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  - Pareto optimal (PO)

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- Goal: EF1 + PO allocation
- Existence?
  - NO [CKMPS14] for general (subadditive) valuations
  - YES for additive valuations [CKMPS14]

submodular valuations

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submodular valuations

EF1+PO (Additive)

- Computation: pseudo-polynomial time algorithm [BKV18]

  

  Complexity of finding an EF1+PO allocation

- Difficulty: Deciding if an allocation is PO is co-NP-hard [KBKZ09]

EF1+PO (Additive)

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  Complexity of finding an EF1+PO allocation

- Difficulty: Deciding if an allocation is PO is co-NP-hard [KBKZ09]

- Approach: Achieve EF1 while maintaining PO
  - PO certificate: competitive equilibrium!

Fairness Notions for Indivisible Items

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Envy-Freeness up to Any Item (EFX) [CKMPS14]

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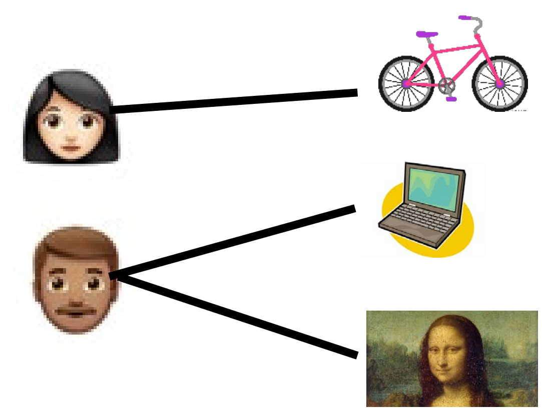
EF1 ?

EFX ?

[15, 10, 20]

[1, 20, 10]



EFX: Existence

- General Valuations [PR18]
  - Identical Valuations
  - $n = 2$

- Additive Valuations
  - $n = 3$ [CGM20]

Additive ($n > 3$), General ($n > 2$)

"Fair division's biggest problem" [P20]

Summary

Covered

- EF1 (existence/polynomial-time algorithm)
- EF1 + PO (partially)
- EFX

Not Covered

- EFX for 3 (additive) agents
- Partial EFX allocations
  - Little Charity [CKMS20]
  - High Nash welfare [CGH19]
- Chores
  - EF1 (existence/ polynomial-time algorithm)

Major Open Questions (additive valuations)

- EF1+PO: Polynomial-time algorithm
- EF1+PO: Existence for chores
- EFX : Existence

References (Divisible Case).

[Aziz20] Aziz, Haris. "The Hylland-Zeckhauser Rule Under Bi-Valued Utilities." *arXiv preprint arXiv:2006.15747* (2020).

[AKT17] Alaei, Saeed, Pooya Jalaly Khalilabadi, and Eva Tardos. "Computing equilibrium in matching markets." *Proceedings of the 2017 ACM Conference on Economics and Computation*. 2017.

[BM04] Anna Bogomolnaia and Herve Moulin. Random matching under dichotomous preferences. Econometrica, 72(1):257–279, 2004.

[BMSY17] Anna Bogomolnaia, Herve Moulin, Fedor Sandomirskiy, and Elena Yanovskaia. Competitive division of a mixed manna. Econometrica, 85(6):1847–1871, 2017.

[BMSY19] Anna Bogomolnaia, Herve Moulin, Fedor Sandomirskiy, and Elena Yanovskaia. Dividing bads under additive utilities. Social Choice and Welfare, 52(3):395–417, 2019.

[BS19] Brânzei, Simina, and Fedor Sandomirskiy. "Algorithms for Competitive Division of Chores." *arXiv preprint arXiv:1907.01766* (2019).

[GM20] Garg, Jugal, and Peter McGlaughlin. "Computing Competitive Equilibria with Mixed Manna." *Proceedings of the 19th International Conference on Autonomous Agents and MultiAgent Systems*. 2020.

[CGMM20] Chaudhury, B. R., Garg, J., McGlaughlin, P., & Mehta, R. (2020). Competitive Allocation of a Mixed Manna. *arXiv preprint arXiv:2008.02753*.

[CGMM20] Chaudhury, B. R., Garg, J., McGlaughlin, P., & Mehta, R. (2020). Dividing Bads is Harder than Dividing Goods: On the Complexity of Fair and Efficient Division of Chores. *arXiv preprint arXiv:2008.00285*.

[DK08] Devanur, Nikhil R., and Ravi Kannan. "Market equilibria in polynomial time for fixed number of goods or agents." *2008 49th Annual IEEE Symposium on Foundations of Computer Science*. IEEE, 2008.

[DPSV08] Devanur, Nikhil R., et al. "Market equilibrium via a primal--dual algorithm for a convex program." *Journal of the ACM (JACM)* 55.5 (2008): 1-18.

[HZ79] Aanund Hylland and Richard Zeckhauser. The efficient allocation of individuals to positions. Journal of Political economy, 87(2):293–314, 1979.

[VY20] Vazirani, Vijay V., and Mihalis Yannakakis. "Computational Complexity of the Hylland-Zeckhauser Scheme for One-Sided Matching Markets." *arXiv preprint arXiv:2004.01348* (2020).

References (Indivisible Case).

[BKV18] Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. Finding fair and efficient allocations. In: *EC 2018*

[B11] Eric Budish. "The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes". In: *J. Political Economy* 119.6 (2011)

[CKMPSW14] Ioannis Caragiannis, David Kurokawa, Herve Moulin, Ariel Procaccia, Nisarg Shah, and Junxing Wang. "The Unreasonable Fairness of Maximum Nash Welfare". In: *EC* 2016

[CGH20] Ioannis Caragiannis, Nick Gravin, and Xin Huang. Envy-freeness up to any item with high Nash welfare: The virtue of donating items. In: *EC 2019*

[CGM20] Bhaskar Ray Chaudhury, Jugal Garg, Kurt Mehlhorn: EFX Exists for Three Agents**.** In: *EC 2020*

[CKMS20] Bhaskar Ray Chaudhury, Telikepalli Kavitha, Kurt Mehlhorn, and Alkmini Sgouritsa. A little charity guarantees almost envy-freeness. In: *SODA 2020*

[KBKZ09] Bart de Keijzer, Sylvain Bouveret, Tomas Klos, and Yingqian Zhang. "On the Complexity of Efficiency and Envy-Freeness in Fair Division of Indivisible Goods with Additive Preferences". In: *Algorithmic Decision Theory (ADT)*. 2009

[LMMS04] Richard J. Lipton, Evangelos Markakis, Elchanan Mossel, and Amin Saberi. "On approximately fair allocations of indivisible goods". In: *EC 2004*

[PR18] Benjamin Plaut and Tim Roughgarden. Almost envy-freeness with general valuations. In: *SODA 2018*

[P20] Ariel Procaccia: An answer to fair division's most enigmatic question: technical perspective. In: *Commun. ACM 63(4): 118 (2020)*