**Lastlec:**

**AwesomeAuctions:**

1. DSIC (agents are truthful)
2. Outcome (allocation) is sociallywelfare maximize
3. Polytime-computable.

**Today:** Single Parameter Auctions (aim: awesome auction)

Single thing/stuff on auction. E.g., item, clicks, bandwidth,

\( v_i \): value $88$ bidder $i$ / unit $88$ to sell.

**Format:**

1. Bidder \( i \) submits \( b_i(\pm v_i) \) in a sealed bid.
2. Seller decides allocation: s.w. maximizing \( v_i \) (dearable allocation)
3. """"payoff \( \to \) Mediation \( \to \) Allocation \( (x_i(b))_{i \in [n]} \) \( x \)

\( b_i, \ldots , b_n \) \( \to \) Mediation \( \to \) Payment.

\( u_i(b) = v_i x_i(b) - p_i(b) \)

Individual: \( u_i(b) \geq 0 \)

Rational: \( 0 \leq p_i(b) \leq b_i x_i(b) \)

Socialwelfare-max allocation

\[ \arg\max \sum_{i=1}^{n} v_i x_i \]
\[ \Rightarrow 0 \leq \frac{1}{p_i(b_i)} \]
\[ b_i = 0 \Rightarrow p_i(b_i) = 0 \]

*Feasible Allocations Examples:*

\[ X \]

S.W. maximizes allocation \((b_1, \ldots, b_n)\) Single item.
\[ \{ x \mid x \in \{0, 1\}^n, \sum x_i = 1 \} \]

\[ x^* \]

\[ B = \max \text{ bid } b_i \]
\[ b_1 \geq b_2 \geq \ldots \geq b_k \text{ highest bid in } b_i \]

\[ b = b_1 \text{ highest bid in } b_i \]

\[ b_{k+1} \text{ if } i \leq K \]

\[ x_i \]

\[ \sum x_i \leq K \]

\[ X \]

\[ k \text{-identical items} \]

\[ \{ x \mid x_i \in \{0, 1\}, \sum x_i = k \} \]

\[ x_i \geq x \]

\[ x_i \leq k \]

\[ \{ x \mid x_i \in \{0, x_i, \ldots, x_k\} \text{ at least one } x_i = x \text{ if } k \} \]

\[ \overrightarrow{b_1} \]

\[ \overrightarrow{\ldots} \]

\[ \overrightarrow{b_k} \]

\[ \overrightarrow{b_{k+1}} \]

**Def 1:** An allocation rule is "Implementable", if it is payoff rule \( p(\cdot) \) s.t. \((x, p)\) is DSIC mechanism.

**Def 2:** An allocation rule is "Monotone" if \( x_i \cdot b_i \) is monotone w.r.t. \( b_i \).
Myerson's Lemma:

1. \( x(\cdot) \) is implementable if and only if it is monotone.
2. If \( x(\cdot) \) is monotone then there exists a unique payout rule \( p(\cdot) \) s.t. \( (x, p) \) is DSIC.
3. There is an explicit formula for \( p(\cdot) \).

Claim 1: If \( (x, p) \) is DSIC \( \Rightarrow x(\cdot) \) is monotone.

Proof:
Fix \( i, b_i \). Now on \( x(b_i) = x_i(b_i, b_{-i}) \) is monotone \( p(b_i) = p_i(b_i, b_{-i}) \).

\( \text{(Recall, } v_i(b) = v; x(b_i) - p(b_i)) \)

\[ z \neq y \text{ two nos.} \]

\[ \text{DSIC } \Rightarrow v_i = b_i = z, \quad v_i = z, b_i = y \]

\[ z \cdot x(z) - p(z) \geq z \cdot x(y) - p(y) \Rightarrow (1) \]

\[ v_i = b_i = y, \quad v_i = y, b_i = z \]

\[ y \cdot x(y) - p(y) \geq y \cdot x(z) - p(z) \Rightarrow (2) \]

\[ z \cdot (x(y) - x(z)) \leq p(y) - p(z) \leq y \cdot (x(y) - x(z)) \Rightarrow (3) \]

\[ (y - z) \cdot (x(y) - x(z)) \geq 0 \]

\[ y > z \Rightarrow x(y) > x(z) \]

\( \text{increasing} \)
Claim 2: \( x_i(\cdot) \) is monotone \( \Rightarrow \) 3 unique payment \( p_i(\cdot) \) formula.

\[
\begin{align*}
\text{Case I: If } z & \text{ is in } [y \rightarrow z] \Rightarrow x(y) = x(z) \text{ as } y \rightarrow z \\
0 & \leq p(y) - p(z) \leq 0 \Rightarrow p(y) = p(z) \\
\text{Case II: If } z & \text{ is at a break point.}
\end{align*}
\]

\[
\begin{align*}
p(y) & = p(z) + z \cdot h \\
p(b_i) & = p(b_i) - p(z) + p(z_d) - p(z_{d-1}) = \text{hd} \cdot z_d \\
& + p(z_1) - p(0) = h_i \cdot z_i \\
& + p(0) = 0
\end{align*}
\]

\[
p(b_i) = \sum_{k=1}^{d} \delta_k \cdot j_{i,p} \text{ in } x(\cdot) \text{ at } x_i
\]
Claim 3: \((x, P)\) is DSIC.

**Proof:** (by Picture.)
Vickey: single item

\[ b_1 \geq b_2 \geq \ldots \geq b_n \]

\( x \) winner payment. \( i = 1 \)

\( B = \max \) bid in

\[ b_1, b_2, \ldots, b_n \]

\[ i = b_2 \]

\[ P_i(b_i) = B = b_2 \]

- Myerson's payment.
- Vickrey payment.

\[ \sum_{d \neq i} \left( x_d - x_{d+1} \right) \]

Sponsored Search: \( k \)-slots

\( a_1 > a_2 > \ldots > a_k > 0 \)

\( a_{k+1} \)

\( b_1 > b_2 > \ldots > b_k > b_{k+1} \geq \ldots \geq b_n \)

\( a_{i(i)} \) goes to slot \( i \)

\[ \max \in a_{i(i)} \cdot x_i \]

\[ P_i(b_i) = b_{k+1} (a_k - 0) + b_{k} (a_{k-1} - a_k) + b_{k-1} (a_{k-2} - a_{k-1}) + \ldots \]

\[ = \sum_{d \neq i} \left( x_d - x_{d+1} \right) \]
\[ \Pi_i(b_i) = \text{Externality caused by agent } i \text{ to the welfare of the system} \]

= Welfare of all the other agents when \( i \) does not participate in the auction

- Welfare of all the other agents when \( i \) participates in the auction