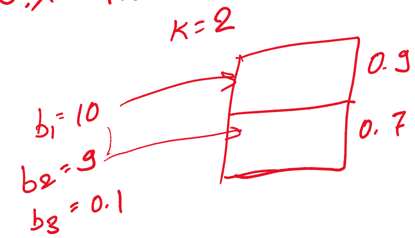
# Single Parameter Auctions: Myerson's Lemma

Thursday, October 29, 2020 1:56 PM

**Last lec:** Single item → First price → NE
→ second price → DSIC (truthful) (Awesome)

Sponsored search → ②, ③. does it satisfy ①? ✗ Not truthful.




$0.9 \times (10-9) < 0.7(9-0.1)$

## Awesome Auctions:

① DSIC (≡ agents are truthful)

② Outcome (allocation) is social welfare maximize

③ Polytime - computable.

## ☆ Today: Single Parameter Auctions (aim: awesome auction)

Single thing / stuff on auction. Eg., item, clicks, bandwidth,

$v_i$ : value of bidder $i$ / unit of the stuff.

### Format:

① Bidder $i$ submits $b_i (\neq v_i)$ in a sealed bid.

② Seller decides feasible allocation: s.w. maximizing wrt bids ($X$: feasible allocations) → **Mechanism**

③ " " payment → **Mechanism**

$(b_1, \dots, b_n) \to$ **Mechanism** $\to$ Allocation $(x_i(b))_{i \in [n]} \in X$

→ Payment. $P_i(b)$ $\forall i$

**Social welfare-max allocation**

$$\underset{x \in X}{\arg\max} \sum_{i=1}^{n} \overset{v_i}{b_i}\, x_i$$

$$U_i(b) = v_i\, x_i(b) - P_i(b)$$

Individual Rational: $U_i(b) \geq 0$

$\Leftarrow \quad 0 \leq P_i(b) \leq b_i\, x_i(b)$

$P_i(b) = 0$

$\Leftarrow \quad 0 \leq p_i(b) - \ldots$

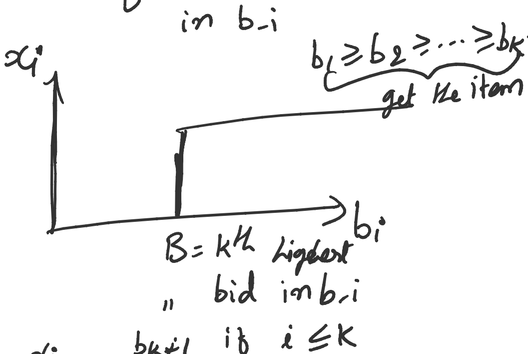
$b_i = 0 \Rightarrow p_i(b) = 0$

## ★ Feasible Allocations Examples : $(X)$

S.W. maximizing allocation $(b_1, \ldots, b_n)$


Single item.

$$\left\{ x \;\middle|\; x_i \in \{0,1\},\ \sum_{i=1}^{n} x_i \leq 1 \right\}$$

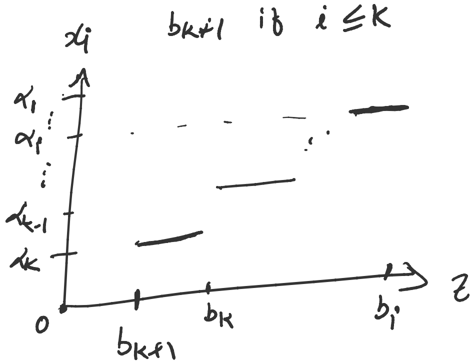


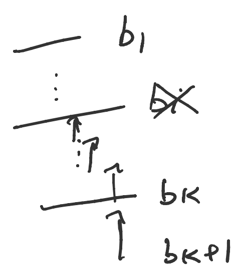

K-identical items.

$$\left\{ x \;\middle|\; x_i \in \{0,1\},\ \sum_{i=1}^{n} x_i \leq K \right\}$$

sponsored search (k-slots)

$$\left\{ x \;\middle|\; \begin{array}{l} x_i \in \{0, \alpha_1, \ldots, \alpha_k\} \\ \text{at most one } x_i = \alpha_j \quad \forall j \leq k \end{array} \right\}$$




**Def 1:** An allocation rule $x(\cdot)$ is "Implementable", if $\exists$ payment rule $p(\cdot)$ s.t. $(x, p)$ is DSIC mechanism.

**Def 2:** An allocation rule is "Monotone" if $\forall i$, $\forall b_{-i}$ $x_i(b_i, b_{-i})$ is monotone wrt $b_i$.

## Myerson's Lemma:

1. $x(\cdot)$ is implementable iff it is monotone.
2. If $x(\cdot)$ is monotone then $\exists$ a unique payment rule $p(\cdot)$ s.t. $(x, p)$ is DSIS
3. There is an explicit formula for $p(\cdot)$.

---

**Claim 1**: If $(x, p)$ is DSIC $\Rightarrow$ $x(\cdot)$ is monotone.

**Pf**: Fix $i$, $b_{-i}$. Now on $x(b_i) = x_i(b_i, b_{-i})$ is monotone

$p(b_i) = p_i(b_i, b_{-i})$

(Recall, $u_i(b) = v_i\, x(b_i) - p(b_i)$)

$z \neq y$ two nos.

DSIC $\Rightarrow$ $v_i = b_i = z$ $\qquad v_i = z \;\; b_i = y$

$$z\,x(z) - p(z) \geq z\,x(y) - p(y) \rightarrow (1)$$

$v_i = b_i = y \qquad v_i = y, \; b_i = z$

$$y\,x(y) - p(y) \geq y\,x(z) - p(z) \rightarrow (2)$$

$$z\,(x(y) - x(z)) \overset{(1)}{\leq} p(y) - p(z) \overset{(2)}{\leq} y\,(x(y) - x(z)) \rightarrow (*)$$

$$\Rightarrow (y - z)(x(y) - x(z)) \geq 0$$

$$\Downarrow$$

$$y > z \Rightarrow x(y) \geq x(z)$$

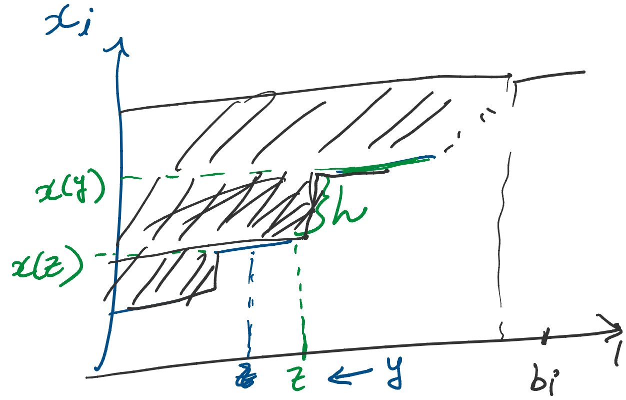
$\quad$ || $\qquad$ increasing

$y > z \Rightarrow x(y) \ldots$

$\Downarrow$

$x$ is monotone increasing

Claim 2: $x_i(\cdot)$ is monotone $\Rightarrow$ $\exists$ unique payment $p_i(\cdot)$ formula.




Pf: $z \neq y$

(*) $z(x(y)-x(z)) \le p(y)-p(z) \le y(x(y)-x(z))$

$y \downarrow z$

Case I: If $z$ is in bet$^n$ useg

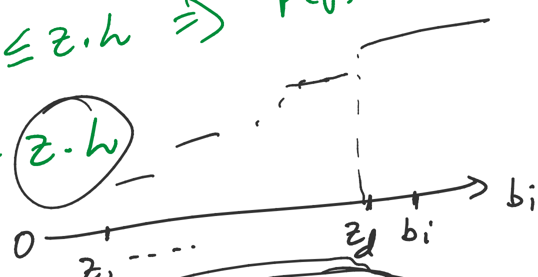
$x(y) = x(z)$ as $y \to z$

(*) $\Rightarrow$ $0 \le p(y)-p(z) \le 0 \Rightarrow p(y)=p(z)$

Case II: If $z$ is at a break point. $p(y)-p(z) = z\cdot h$

$z\cdot h \le p(y)-p(z) \le z\cdot h \Rightarrow$

$p(y) = p(z) + z\cdot h$



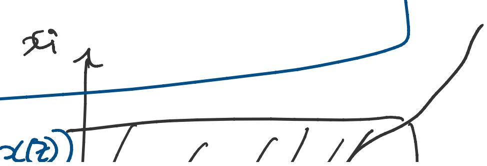

$$P(b_i) = \underbrace{(p(b_i) - P(z_d))}_{=0} + (P(z_d) - P(z_{d-1})) = h_d \cdot z_d$$

$$+ \;\vdots$$

$$+ (P(z_1) - P(0)) = h_1 z_1$$

$$+ P(0) = 0$$

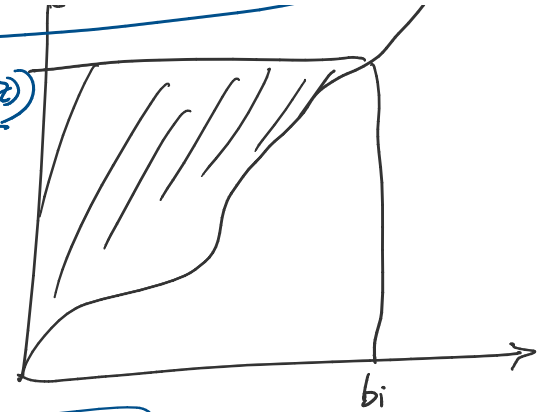
$$\boxed{p(b_i) = \sum_{k=1}^{d} z_k \cdot \text{jump in } x(\cdot) \text{ at } z_k}$$




$\ldots \le y(x(y)-x(z))$

$k=1$

$$\lim_{y \to z} \frac{z\,(x(y) - x(z))}{y-z} \le \frac{P(y) - P(z)}{y-z} \le \frac{y\,(x(y)-x(z))}{y-z}$$




$$= z\, x'(z) \le P'(z) \le z \cdot x'(z)$$

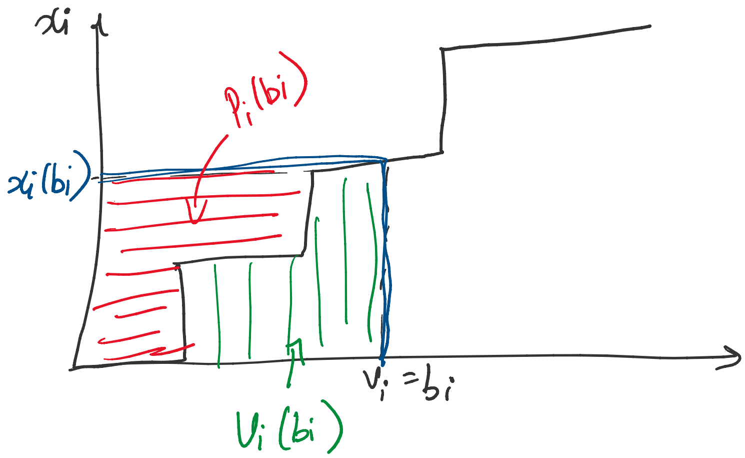
$$\Rightarrow P'(z) = z \cdot x'(z)$$

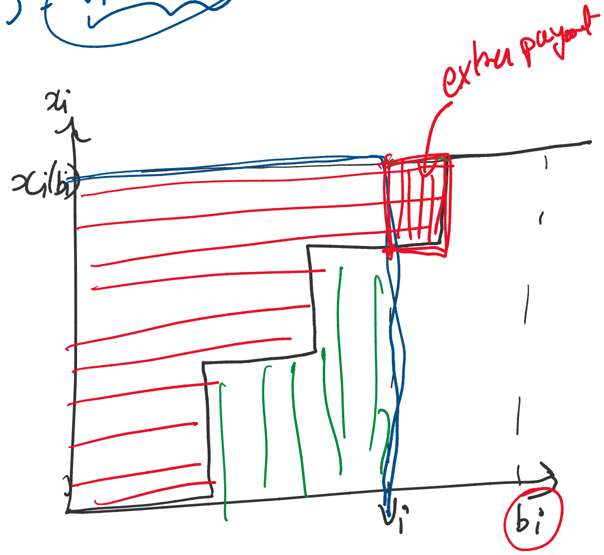
$$\Rightarrow \boxed{P(b_i) = \int_0^{b_i} P'(z)\,dz = \int_0^{b_i} z\, x'(z)\, dz}$$

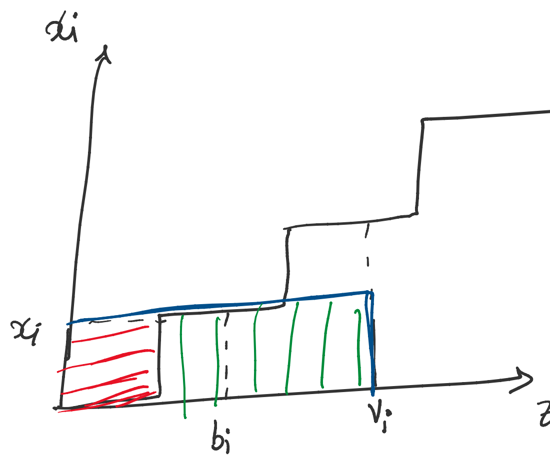
Claim 3: $(x, P)$ is DSIC.

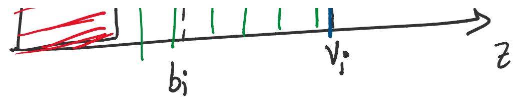
$$U_i(b_i) = \underbrace{v_i\, x_i(b_i)} - P_i(b_i)$$

Pf: (by Picture).



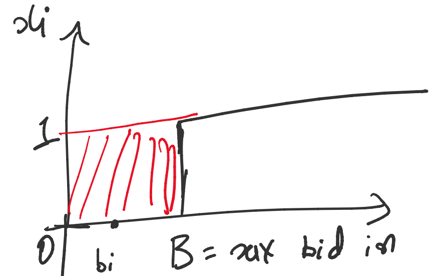



Vickery : Single item

$$\textcircled{b_1} \geq \textcircled{b_2} \geq \dots \geq b_n$$

winner    payment.

$i = 1$




$B$ = max bid in $b_{-i} = (b_2, \dots, b_n)$

$= b_2$

$P_1(b_1) = B = b_2$

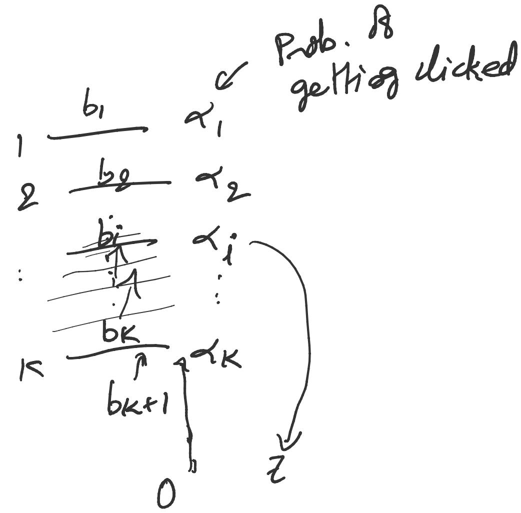
= Myerson's payment.

= Vickery payment.

Sponsored Search : $k$-slots

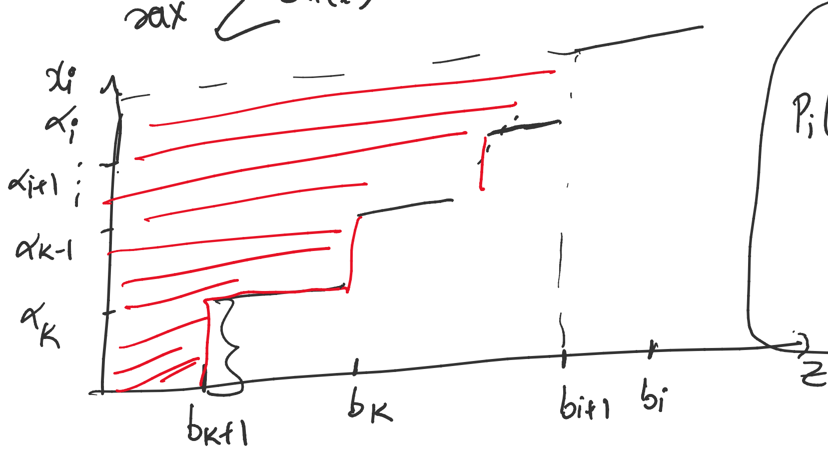
$$\alpha_1 > \alpha_2 > \dots > \alpha_k > \underset{\alpha_{k+1}}{\overset{\shortparallel}{0}}$$

$$b_1 > b_2 > \dots > b_k > b_{k+1} > \dots > b_n$$




$b_{\pi(i)}$ goes to slot $i$

$$\max \sum b_{\pi(i)} \cdot \alpha_i$$




$$\boxed{\begin{aligned} P_i(b_i) = {} & b_{k+1}(\alpha_k - 0) \\ & + b_k(\alpha_{k-1} - \alpha_k) \\ & \vdots \\ & + b_{i+1}(\alpha_i - \alpha_{i+1}) \end{aligned}}$$

$$= \sum_{d=i}^{k} b_{d+1}(\alpha_d - \alpha_{d+1})$$


$$= \sum_{d=k}^{i} b_{d+1}(\alpha_d - \alpha_{d+1})$$

$p_i(b_i)$ = Externality caused by agent $i$ to the welfare of the system

= Welfare of all the other agents when $i$ does not participate in the auction.

−

Welfare of all the other agents when $i$ participates in the auction