Designing rules "as the game" s.t. desired outcome is achieved.

Auctions
- seller/auctioneer
- sell single item (indivisible)
- set 3 as agents/bidders.
  \[ v_i = \text{value of bidder } i \text{ for the item}. \]

What if \( v_i \)'s are known?

\[ \text{Price} = \max_i v_i - c \]

Issue: \( v_i \) is a "private information" as agent \( i \).

Sealed Bid Auction:
1. Bidders bid: \( b_i \) is bid for agent \( i \) \( (b_i \text{ need not be}) \)
   in a sealed envelope

2. Auctioneer looks at all the bids & decides the
   winner/allocation
   winner = highest bidder: \( \max_i b_i \)

3. \( k \) what the winner pays. \( p \).

\[ \text{utility} (i) = U_i = v_i - p \text{ if } i \text{ is a winner} \]
\[ = 0 \text{ o.w.} \]

\[ \times \]
\[ h = 0 \text{ (pay nothing)} \times \]

First-price: pay your bid
\( \times \) is the winner \( \implies p = b_i \)
\( \times \) bid

\[ \begin{array}{c|ccc}
  & v_1 & v_2 & v_3 \\
  \hline
  p_1 & 200 & 600 & 500 \\
  p_2 & 150 & 375 & 400 \\
  p_3 & 200 & 465 & 450 \\
\end{array} \]
First-Price:

2-bidders \( v_1, v_2 \sim U[0, 1] \) (Private Information)

Suppose, \( b_2 = \frac{v_2}{2} \)

If bidder 1 bids \( b_1 \)

\[
\alpha \times \frac{v_1 - b_1}{b_1} \Pr[b_2 \leq b_1] + \alpha \times \frac{v_1 - b_1}{b_1} \Pr[v_2 \leq 2b_1]
\]

\[
\alpha \times \frac{v_1 - b_1}{b_1} \left(\frac{v_1 - b_1}{b_1} \right) \Rightarrow \frac{d}{db_1} (v_1 - 2b_1) = 0
\]

\[
v_1 - 2b_1 = 0 \Rightarrow b_1 = \frac{v_1}{2}
\]

\((\frac{v_1}{2}, \frac{v_2}{2})\) is a NE

Generalization

\(n\)-players. \( v_1, \ldots, v_n \sim U[0, 1] \)

Fix: \( b_k = \frac{(m-D)vk}{n} \) \( v_k+i \)

Agent \( i \) wants to bid \( b_i \) s.t.

\[
\alpha \times \frac{v_i - b_i}{b_i} \Pr[b_k \leq b_i, v_k+i] \]

Second-price: pay second highest bid

\( i^* \) pays \( p = \alpha x_i b_i \)
$\sum_{i} \max_{b_i} (V_i - b_i) = \frac{\sum_{i} V_i}{n}$

Bidding $\frac{\sum_{i} V_i}{n}$ is Bayesian NE

$\Rightarrow$ what is arbitrary distribution?
$\Rightarrow$ what if arbitrary distributions?

Very complex. Hard reason about.

Second-price

winner $i^* = \max_{i} b_i \quad b_i \geq b_{i+1} \Rightarrow \cdots \Rightarrow b_n$

payoff $p = \max_{i \neq i^*} b_i \quad \rightarrow \text{ winner payoff.}$

[Vickery '61]: (Bidding your true value is the best strategy.)

For every agent $i$, $b_i = V_i$ is dominant strategy

no matter what $b_i$ is. ($\equiv b_i = V_i$ is DSNE)

Pf: Fix an agent $i$. Two possible outcomes.

- $i$ wins $\Rightarrow V_i = V_i - \max_{k \neq i} b_k = B$

- $i$ loses $\Rightarrow V_i = 0$ ($b_i \leq \max_{k \neq i} b_k = B$)

(Payment of the winner is independent of his bid)

Case I: $V_i \leq B \Rightarrow b_i = V_i$ is the best

\[
\text{Utility} \quad \begin{cases} 
0 & \text{if } V_i < B \\
V_i - B & \text{if } V_i = B \\
B - V_i & \text{if } V_i > B 
\end{cases}
\]
Bidding your true value is dominant strategy.

 Dominant strategy Incentive compatible. (DSIC)

1. DSIC = truthful reporting.
2. Social welfare maximizing allocation.
3. Polynomial-time computable (winner, payoffs)

Awesome Auctions.

Vickey is an awesome auction.

English Auction (open-bid auctions): (Ebay auction)

- Auctioneer announces a very low price 0.
- There is a set of buyers willing to buy at the current price.
- Keep increasing price until one bidder remains.

\[ v_1 \geq v_2 \geq \ldots \geq v_n \geq 0 \]

\[ \text{Price} = 0 \quad \uparrow \]

\[ a_i \cdot v = v_0 + \epsilon \]
Second-Price auction:

- Dutch Auction:
  - Auctioneer starts with very high price
  - If no one wants to pay,
  - Then keep decreasing until at least one agent is willing to pay.

\[
\begin{align*}
v_1 \geq v_2 \geq \cdots \geq v_m
\end{align*}
\]

\[
\text{Price} > p \rightarrow 
\]

\[
p = v_1
\]

\[\text{Close to first-price auction.}\]

Search Auction: