Smooth Games (Contd.)

Tuesday, October 20, 2020 1:48 PM

P: NE , P^*: OPT $\qquad PoA = \dfrac{cost(P)}{cost(P^*)}$

① NE $\Rightarrow$ $C_i(P) \le C_i(P_i^*, P_{-i})$, $\forall i$

$$cost(P) = \sum_{i=1}^{n} C_i(P) \le \sum_{i=1}^{n} C_i(P_i^*, P_{-i})$$

$$\boxed{② \quad \sum_{i=1}^{n} C_i(P_i^*, P_{-i}) \le \underbrace{\frac{5}{3}}_{\to\lambda} cost(P^*) + \underbrace{\frac{1}{3}}_{\to\mu} cost(P)}$$

③ $\left(1 - \frac{1}{3}\right) cost(P) \le \frac{5}{3} cost(P^*) \Rightarrow PoA = \dfrac{cost(P)}{cost(P^*)} \le \dfrac{5/3}{1-1/3}$

$$= \frac{\lambda}{1-\mu}$$

★ Smooth Game:

Game w/ $\{1 \ldots n\}$. Player i's pure actions S_i

$\forall (s_1, \ldots, s_n) \in \bigtimes_{i=1}^{n} S_i = S$. $(s_1^*, \ldots, s_n^*) \in S$

$$\sum_{i=1}^{n} C_i(s_i^*, s_{-i}) \le \lambda \, cost(s^*) + \mu \, cost(s)$$

$$\boxed{\text{Pure NE} \subseteq \text{Mixed NE} \subseteq \text{Correlated eq. (CE)} \subseteq \text{Coarse CE (CCE)}}$$

$$PoA^{\underline{PNE}} \le PoA^{MNE} \le PoA^{CE} \le PoA^{\underline{CCE}} \le \frac{\lambda}{1-\mu}$$

★ ϵ-CCE: $\underline{\sigma} \sim S$ (joint distribution on all possible outcomes of the game).

$$\forall i: \ \mathbb{E}_{s\sim\sigma}[C_i(s)] \le (1+\epsilon) \mathbb{E}_{s\sim\sigma}[C_i(s_i', s_{-i})] \quad \forall s_i' \in S_i$$

$$\boxed{cost(\sigma) = \mathbb{E}_{s\sim\sigma}[cost(s)] = \mathbb{E}_{s\sim\sigma}\left[\sum_i C_i(s)\right]}$$

$$\boxed{\text{cost}(\sigma) = \mathbb{E}_{s\sim\sigma}[\text{cost}(s)] = \sum_{s\sim\sigma} \cdots}$$

$$\text{PoA}^{CCE} = \max_{\sigma \in CCE} \frac{\text{cost}(\sigma)}{\text{cost}(s^*)} \qquad s^*: \text{OPT}.$$

Thm: $\text{PoA}^{CCE} \le \dfrac{\lambda}{1-\mu}$ for smooth games.

Pf: σ: max cost ~~ε-~~CCE. $\quad s^*$: OPT

$$\text{CCE} \Rightarrow \underline{\sum_i \mathbb{E}_{s\sim\sigma}[C_i(s)]} \le (1+\epsilon)\sum_i \mathbb{E}_{s\sim\sigma}\left[C_i(s_i^*, s_{-i})\right] \qquad : \cancel{\forall i}$$

$$\boxed{\mathbb{E}[X] = \sum_{v\in \text{supp}(X)} v \cdot \Pr(v)}$$

$$\Rightarrow \underline{\text{cost}(\sigma)} \le (1+\epsilon)\sum_i \mathbb{E}_{s\sim\sigma}\left[C_i(s_i^*, s_{-i})\right]$$

$$= (1+\epsilon)\,\mathbb{E}_{s\sim\sigma}\, \underbrace{\sum_i C_i(s_i^*, s_{-i})}$$

$$(\because \text{smooth game}) \le (1+\epsilon)\,\mathbb{E}_{s\sim\sigma}\left(\lambda\, \underline{\text{cost}(s^*)} + \mu\, \text{cost}(s)\right)$$

$$= (1+\epsilon)\lambda\, \text{cost}(s^*) + (1+\epsilon)\mu\, \underbrace{\mathbb{E}_{s\sim\sigma}[\text{cost}(s)]}_{\text{cost}(\sigma)}$$

$$= (1+\epsilon)\lambda\, \text{cost}(s^*) + (1+\epsilon)\mu\, \underline{\text{cost}(\sigma)}$$

$$\Rightarrow (1-\mu)\,\text{cost}(\sigma) \le \lambda\, \text{cost}(s^*)$$

$$\Rightarrow \text{PoA}^{CCE} = \frac{\text{cost}(\sigma)}{\text{cost}(s^*)} \le \frac{\lambda(1+\epsilon)}{1-\mu(1+\epsilon)} \quad \blacksquare$$

Games w/ Positive Externalities:

Network cost-sharing (formation) Games:

- $G = (V, E)$ directed.
- Set of players $\{1 \dots k\}$. i^{th} player wants to "build" a path from s_i to t_i. $s_i, t_i \in V$.

$\mathcal{P}_i$ = s_i to t_i paths. $\quad P_i \in \mathcal{P}_i$

- Cost of building edge e is γ_e.

$P = (P_1, \dots, P_n) \rightarrow f_e$ = # players wanting to build edge e as per P.

$$C_i(P) = \sum_{e \in P_i} \frac{\gamma_e}{f_e}$$

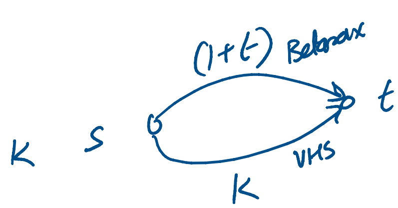
$$Cost(P) = \sum_i C_i(P) = \sum_{e : f_e \geq 1} \gamma_e$$

Ex: (Mis-coordination)

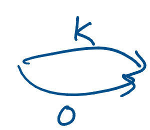
in 1980's VHS vs Betamax

VHS ↑ earlier player

Betamax ↑ Better technology

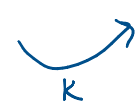


OPT = NE1 :



$\rightarrow$ cost = $(1+\epsilon)$

NE2:



$\rightarrow$ cost = K

$$PoA = \frac{\text{worst NE}}{\text{OPT}} = \frac{K}{1+\epsilon} \sim K = \text{\# players}.$$

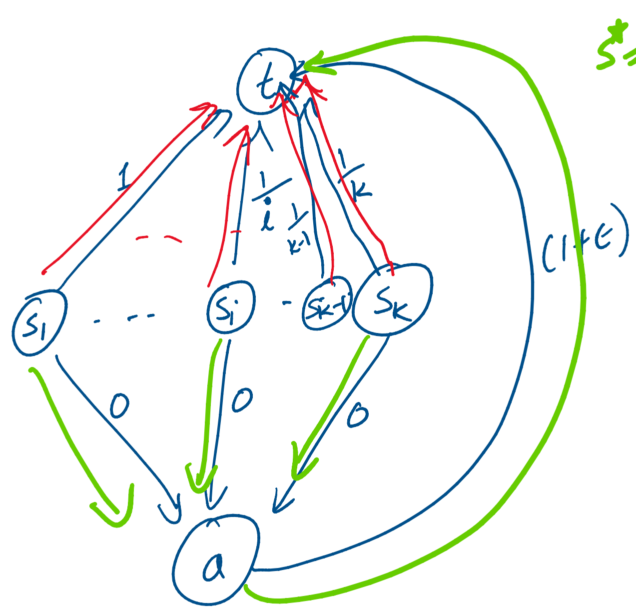
$\Rightarrow$ Introduce some kind early co-ordination through ..diator/default option.

⇒ Introduce some kind early ...
mediator/default option.

$$\text{Price-of-Stability} = \frac{\text{Cost at "best" NE}}{\text{OPT cost}}$$

$$= \min_{s \in NE} \frac{\text{cost}(s)}{\text{cost}(s^*)}$$ ↪ OPT

Eg 2 :



s^* = OPT cost $= (1+\epsilon)$

$c_i(s^*) = \frac{1+\epsilon}{k} \; \forall i$

$c_i((s_k - t), s^*_{-k}) = \frac{1+\epsilon}{k-1} \; \forall i < k$

s: NE (only NE)

$\text{cost}(s) = \sum_{i=1}^{k} \frac{1}{i} \sim \ln k = H_k$

PoS $= H_k$

★ Existence of Pure NE

Cost-sharing Game → Routing game framework..

$$\boxed{c_e(i) = \frac{r_e}{i}}$$ ↓

Rosenthal's Potential func $\phi(P) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$

$$= \sum_{e \in E} r_e \sum_{i=1}^{f_e} \frac{1}{i} \le H_k \quad (\because f_e \le k)$$

$c_i(P_i', P_{-i}) - c_i(P) = \phi(P_i', P_{-i}) - \phi(P)$

$$\boxed{\begin{gathered} c_i(P_i', P_{-i}) - c_i(P) = \phi(P_i', P_{-i}) - \phi(P) \\ \forall i,\ \forall P \in \mathcal{P},\ \forall P_i' \in \mathcal{P}_i \end{gathered}}$$

Thm: Cost-Sharing Games, $PoS \leq H_k$ w/ k players.

Pf:

$$\text{cost}(P) = \sum_{\substack{e \in E \\ f_e \geq 1}} r_e \leq \phi(P) = \sum_{e \in E} r_e \underbrace{\sum_{i=1}^{f_e} \frac{1}{i}}_{\leq H_k} \leq H_k \sum_{\substack{e \in E \\ f_e \geq 1}} r_e = H_k \,\text{cost}(P) \longrightarrow (1)$$

$$\Rightarrow \text{cost}(P) \leq \phi(P) \leq H_k \,\text{cost}(P).$$

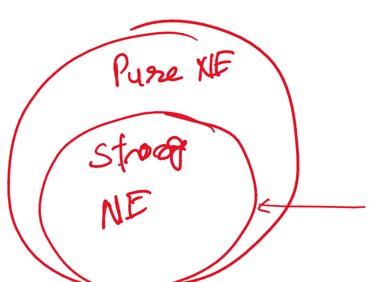
$\tilde{P} \in \arg\min_{P \in \mathcal{P}} \phi(P) \Rightarrow \tilde{P}$ is a NE..

P^*: OPT, $P^* = \arg\min_{P \in \mathcal{P}} \text{cost}(P)$

$$\text{cost}(\tilde{P}) \leq \phi(\tilde{P}) \leq \phi(P^*) \leq H_k \,\text{cost}(P^*) \qquad (\because (1))$$

$$\Rightarrow PoS \leq \frac{\text{cost}(\tilde{P})}{\text{cost}(P^*)} \leq H_k \qquad \blacksquare$$

☆



"Beneficial Deviation": $\exists A \subseteq \{1, \ldots, k\}$ (coalition)

$s_A' \in \times_{i \in A} S_i$

"Benefitial ..."

$$s_A' \in \bigtimes_{i \in A} S_i'$$

$\forall i \in A$: $\quad c_i(s_A', s_{-A}) \le c_i(s)$

Strict inequality for at least one $i \in A$.

* S is Strong NE if there is no subset of players with benefitial deviation.

Thm: $\quad PoA^{\text{strong NE}} \le H_k$ for k-player cost-sharing games.

Pf: $\quad P$: a strong NE $\qquad P^*$: opt.

- Why $A_k = \{1, \dots, k\}$ do not wanna move to P^*

because $\exists i \in A_k$ s.t. $c_i(P^*) \ge c_i(P)$. let's call this player k.

- why $A_{k-1} = \{1, \dots, k-1\}$ do not wanna move to P^*

$\exists i \in A_{k-1}$ s.t. $c_i(P^*_{A_{k-1}}, P_k) \ge c_i(P)$, let's call this player $(k-1)$

$\vdots$

$A_\ell = \{1, \dots \ell\}$ $\quad \dots \quad P^*$

$\exists i \in A_\ell$ s.t. $\underline{c_i(P^*_{A_\ell}, P_{-A_\ell}) \ge c_i(P)}$, .. call this player ℓ.

$\vdots$

$$\text{cost}(P) = \sum_{i=1}^{k} c_i(P) \le \sum_{i=1}^{k} c_i(P^*_{A_i}, P_{-A_i}) \quad (\text{here } A_i = \{1, \dots, i\})$$

$$\le \sum_{i=1}^{k} c_i(P^*_{A_i}) \rightarrow ①$$

$$\leq \sum_{i=1} c_i(P_{A_i})$$

★ $P^*_{A_i}$, $A_i = \{1, \dots i\}$

f^i_e = # players building edge e in $P^*_{A_i}$

$= |\{l \in A_i \mid e \in P^*_l\}|$

$$c_i(P^*_{A_i}) = \sum_{e \in P^*_l} \frac{r_e}{f^i_e} \overset{?}{=} \phi(P^*_{A_i}) - \phi(P^*_{A_{i-1}})$$

$$① \Rightarrow \quad cost(P) \leq \sum_{i=1}^{K} c_i(P^*_{A_i}) = \sum_{i=1}^{K} \phi(P^*_{A_i}) - \phi(P^*_{A_{i-1}})$$

$A_0 = \{\}$

$$= \cancel{\phi(P^*_{A_1})} - \underset{}{\overset{0}{\overset{\shortparallel}{\phi}}}(P^*_{A_0}) + \cancel{\phi(P^*_{A_2})} - \cancel{\phi(P^*_{A_1})} + \dots$$

$$+ \phi(P^*_{A_K}) - \cancel{\phi(P^*_{A_{K-1}})}$$

$$= \underbrace{\phi(P^*_{A_K})} - \underbrace{\phi(P^*_{A_0})}_{= 0}$$

$$= \phi(P^*)$$

$$\leq H_K \; cost(P^*)$$

$$\Rightarrow PoA^{\text{strong NE}} = \frac{cost(P)}{cost(P^*)} \leq H_K$$