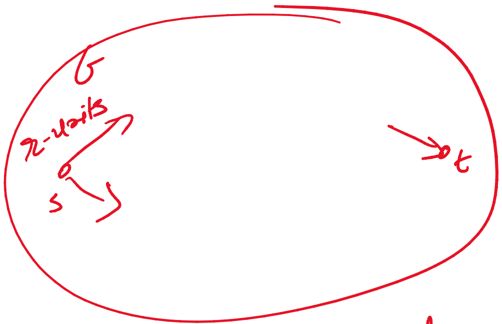
Non-Atomic Routing Games (Contd.)

Tuesday, October 13, 2020 1:53 PM

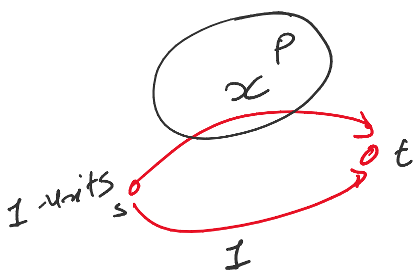
Given: Directed graph $G=(V,E)$. $s,t \in V$. $e \in E$, c_e delay func. non-neg, non-decreasing, continuous.



$$PoA = \frac{\text{worst NE cost}}{\text{OPT cost}} \geq 1$$

Q: How bad the total cost at NE flow can be compared to the OPT.

Pigou

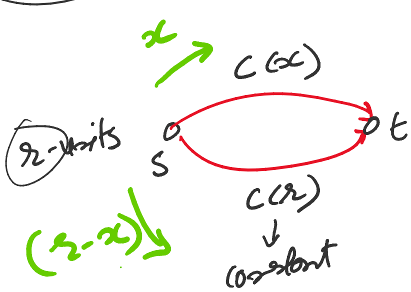


$p \to \infty \qquad PoA \to \infty$

$\mathcal{C}$: class of cost functions.

Thm: PoA for any G with r-units flow $s \to t$ cost $c_e \in \mathcal{C}$ $\leq$ PoA Pigou w/ cost $\mathcal{C}$?

$c \in \mathcal{C}$



PoA

NE: r, 0 $\qquad \text{cost(NE)} = r \cdot c(r)$

$$\text{OPT}: \boxed{\inf_{0 \leq x \leq r} x \cdot c(x) + (r-x)\, c(r)}$$

Claim: $x^* \leq r$, if x^* achieves the inf.

Pf: $c(x) + x \cdot c'(x) - c(r) = 0$

$$P.A = \sup \frac{r \cdot c(r)}{\ldots}$$

Claim: ...

Pf: $c(x) + x \cdot c'(x) - c(r) = 0$

$c(x) + x \cdot c'(x) = c(r)$

$c(x) + \underbrace{x \cdot c'(x)}_{\geq 0}$

$\Rightarrow c(x) \leq c(r) \Rightarrow x \leq r$

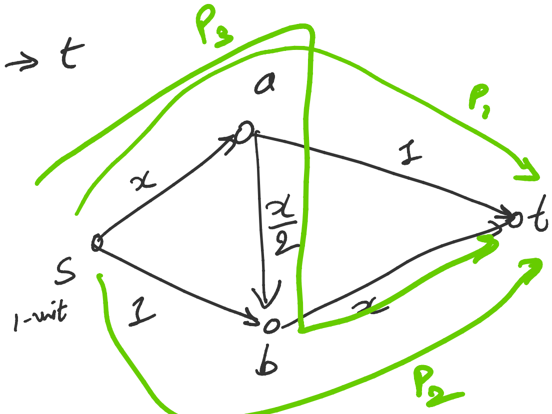
$$PoA = \sup_{x \geq 0} \frac{r \cdot c(r)}{\left(x\, c(x) + (r-x)\, c(r)\right)}$$

$$\alpha(\mathcal{C}) = \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \frac{r\, c(r)}{x \cdot c(x) + (r-x)\, c(r)}$$

Theorem: Given $G = (V, E)$, $s, t \in V$, r-units, $c_e \in \mathcal{C}$ $\forall e \in E$,

$$PoA \leq \alpha(\mathcal{C}).$$

- $\rightarrow$ f: r-units of flow from $s \to t$
- $\rightarrow$ $\mathcal{P}$: set of $s \to t$ paths
- $\rightarrow$ f_p: flow on path p.
- $\rightarrow$ $f_e = \sum_{p : e \in p} f_p$
- $\rightarrow$ Cost on edge e: $c_e(f_e)$
- $\rightarrow$ Cost on path p: $c_p(f) = \sum_{e \in p} c_e(f_e)$



$f_{P_1} = f_{P_2} = \frac{1}{4}$, $f_{P_3} = \frac{1}{2}$

$f_{(s,a)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$f_{(a,t)} = \frac{1}{4}$

$f_{(a,b)} = \frac{1}{2}$ $\quad C_{(a,b)}(f_{(a,b)}) = \frac{1}{4}$

$C_{P_1}(f) = 1 + \frac{3}{4}$

$C_{P_2}(f) = 1 + \frac{3}{4}$

$$\boxed{\begin{array}{l} \text{NE:} \\ \forall p \in \mathcal{P}:\ f_p > 0 \Rightarrow C_p(f) \leq C_q(f) \\ \qquad\qquad\qquad\qquad \forall q \in \mathcal{P} \end{array}}$$

Claim 1: Given any flow f

$$\text{total cost} = \sum_{p \in \mathcal{P}} f_p\, c_p(f) = \sum_{e \in E} f_e\, c_e(f_e)$$

$\leq c_e(f_e)$

follow ... $p \in \mathcal{P}$

Pf:

$$\sum_{p \in \mathcal{P}} f_p c_p(f) = \sum_{p \in \mathcal{P}} f_p \sum_{e \in p} c_e(f_e)$$

$$= \sum_{e \in E} c_e(f_e) \left(\sum_{p:\, e \in p} f_p \right)$$

$$= \sum_{e \in E} c_e(f_e) f_e$$

f: NE flow, f^*: OPT flow.

$cost(f) = \sum_{e \in E} f_e c_e(f_e)$ $cost(f^*) = \sum_{e \in E} f_e^* c_e(f_e^*)$

claim: $\sum_e (f_e^* - f_e) c_e(f_e) \geq 0$

Pf: Fix cost to costs at NE flow f.

$$\boxed{L = \min_{p \in \mathcal{P}} c_p(f)}$$

$$\sum_{p \in \mathcal{P}} f_p c_p(f) = r \cdot L \to ①$$

$$\sum_{p \in \mathcal{P}} f_p^* \underbrace{c_p(f)}_{\geq L} \geq L \cdot \underbrace{\sum_{p \in \mathcal{P}} f_p^*}_{= r} = r \cdot L \to ②$$

$$①, ② \Rightarrow \sum_p f_p^* c_p(f) \geq \sum_p f_p c_p(f)$$

$$\Rightarrow \sum_e f_e^* c_e(f_e) \geq \sum_e f_e c_e(f_e)$$

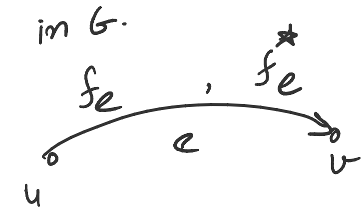
$$\Rightarrow \sum_e f_e^* \, c_e(f_e) \leq \sum_e$$

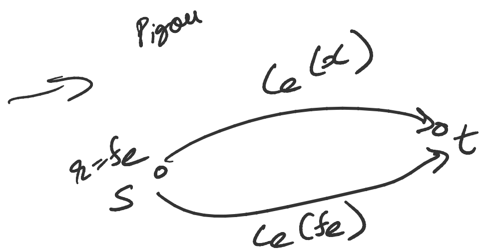
$$PoA = \frac{\sum_e f_e \, c_e(f_e)}{\sum_e f_e^* \, c_e(f_e^*)}$$

Proof of Theorem :

If we prove $\forall e \in E$ $\quad \dfrac{f_e \, c_e(f_e)}{\alpha(e)} \leq f_e^* \, c_e(f_e^*)$ $\quad \times$

$e \in E$.





$$\alpha(e) \geq \frac{r \cdot c(r)}{x \cdot c(x) + (r - x)\, c(r)} = \frac{f_e \cdot c_e(f_e)}{f_e^* \, c_e(f_e^*) + (f_e - f_e^*)\, c_e(f_e)}$$

$$\Rightarrow \sum_e f_e^* \, c_e(f_e^*) \geq \underbrace{\sum_e (f_e^* - f_e)\, c_e(f_e)}_{\geq 0} + \sum_e \frac{f_e \cdot c_e(f_e)}{\alpha(e)}$$

$(\because$ claim 2$)$

$$\geq \frac{\sum_e f_e \, c_e(f_e)}{\alpha(e)}$$

$$PoA = \frac{\sum_e f_e \, c_e(f_e)}{\sum f_e^* \, c_e(f_e^*)} \leq \alpha(e) \qquad \blacksquare$$

$$\overline{\sum_e f_e^* c_e(f_e^*)}$$

∃ n/w with quadratic cost fnc.

$\alpha(\mathcal{C})$ for $\mathcal{C} = \{ax^2 + bx + c \mid a, b, c \geq 0\}$

linear $\mathcal{C} = \{ax + b \mid a, b \geq 0\}$

N/W Over Provisioning:

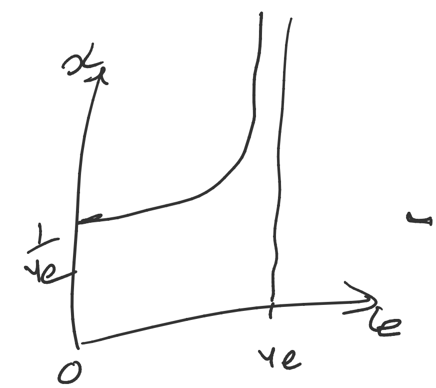
ISP's provide extra capacity in the n/w.

↳ Improves the overall delay cost

↳ Cheaper than smart QoS implementations.

$$c_e(x) = \frac{1}{u_e - x} \quad \text{if } x < u_e$$

$$= \infty \quad \text{if } x \geq u_e$$



(u_e : capacity)

* n/w is α-over provisioned if at NE.

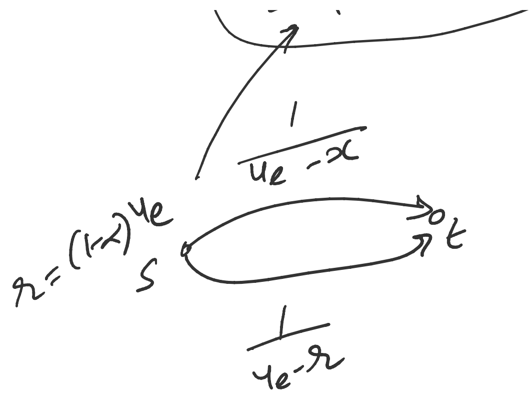
$f_e \leq (1-\alpha) u_e$ (α fraction of edge capacity is unused)

$$PoA \leq \frac{1}{2}\left(1 + \frac{1}{\sqrt{\alpha}}\right)$$

$\alpha = 1 \Rightarrow PoA = 1$

$\alpha \to 0 \Rightarrow PoA \to \infty$

$\alpha = 0.1 \Rightarrow \boxed{PoA \sim 2}$



$\alpha = 0.1$

Theorem: In General graph $G = (V,E)$, for any e, $c_e \in \mathcal{C}$ $\forall e \in E$.

f: NE flow of "r" units
f^*: OPT flow of "$2r$"-units

$\le$ NE flow wrt $c_e(x)$
OPT flow wrt $2c_e(2x)$

(NE $\Downarrow$ w/ capacity u_e
OPT " " $\frac{u_e}{2}$)

$$\text{cost}(f) \le \text{cost}(f^*)$$

$L = \min_p c_p(f)$

$(\because f_p > 0 \Rightarrow c_p(f) = L)$

Pf:

$$\text{cost}(f) = \sum_p f_p c_p(f) = L \cdot \sum_p f_p = L \cdot r$$

$$\text{cost}(f^* \text{ w/ costs at NE}) = \sum_p f_p^* c_p(f) \ge L \cdot \sum_p f_p^* = L \cdot (2r) = 2Lr$$

$$\text{cost}(f) = L \cdot r \le \sum_p f_p^* c_p(f) - L \cdot r \le \text{cost}(f^*)$$

if true then done.

$\sum f_p^* c_p(f^*)$

$$\Rightarrow \sum_p f_p^* c_p(f) - \sum_p f_p c_p(f) \le \sum_p f_p^* c_p(f^*)$$

$$\Updownarrow$$

$$\Rightarrow \sum_e f_e^* c_e(f_e) - \sum_e f_e c_e(f_e) \le \sum_e f_e^* c_e(f_e^*)$$

$$\Updownarrow$$

$$\Rightarrow \sum_e f_e^* \left(c_e(f_e) - c_e(f_e^*) \right) \le \sum_e f_e c_e(f_e)$$

$$\Uparrow$$

$$\Rightarrow \forall e \in E \quad \overbrace{f_e^* \left(c_e(f_e) - c_e(f_e^*) \right)}^{lhs} \le \overbrace{\underline{f_e} c_e(f_e)}^{rhs}$$

case I: $f_e > f_e^*$ $\quad b < a \Rightarrow f_e^* \cdot b \le f_e \cdot a$

$\Rightarrow$ Holds.

case II: $f_e \le f_e^*$ then $c_e(f_e) - c_e(f_e^*) \le 0$

$\Rightarrow lhs \le 0$ & $rhs \ge 0$

$\Rightarrow$ inequality holds. ∎

$$\mathcal{C} = \{ ax + b \mid a, b \ge 0 \}$$

$$\alpha(\mathcal{C}) = \sup_{ax+b \in \mathcal{C}} \sup_{r \ge 0} \sup_{x \ge 0} \overbrace{x(ax+b) + (r-x)(ar+b)}^{r(ar+b)}$$

$$\alpha(\ell) = \sup_{ax+b \in \ell}$$

$$\frac{d}{dx} \quad x(ax+b) + (r-x)(ar+b) = 0$$

$$\Rightarrow 2ax + \cancel{b} - ar - \cancel{b} = 0$$

$$x = \frac{\cancel{a}r}{2\cancel{a}} \Rightarrow \boxed{x = \frac{r}{2}}$$

$$\alpha(\ell) = \sup_{a,b \geq 0} \sup_{r \geq 0} \frac{\cancel{r}(ar+b)}{\frac{\cancel{r}}{2}\left(\frac{ar}{2}+b\right) + \frac{\cancel{r}}{2}(ar+b)}$$

$$= \quad '' \quad '' \quad \frac{ar+b}{ar\left(\frac{1}{4}+\frac{1}{2}\right) + b}$$

$$= \quad '' \quad \lim_{r \to \infty} \quad ''$$

$$= \sup_{a,b \geq 0} \lim_{r \to \infty} \frac{a + \frac{b}{r}^{\to 0}}{a\left(\frac{3}{4}\right) + \frac{b}{r}^{\to 0}}$$

$$= \sup_{a \geq 0} \frac{\cancel{a}}{\cancel{a}\left(\frac{3}{4}\right)}$$

$$= \frac{4}{3}$$