Lecture 3: Computation of CE

CS 598RM

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(Recall) Fisher's Model

- Set A of n agents. Set G of m divisible goods.

- Each agent i has
  - □ budget of B_i dollars
  - □ valuation function $v_i: R_+^m \rightarrow R_+$ over bundles of goods.

  Linear: for bundle $x_i = (x_{i1}, \ldots, x_{im})$,
  $$v_i(x_i) = \sum_{j \in G} v_{ij} x_{ij}$$

- Supply of every good is one.

(Recall) Competitive Equilibrium

Pirces $p = (p_1, \ldots, p_m)$ and allocation $X = (x_1, \ldots, x_n)$

- **Optimal bundle:** Agent i demands
$$x_i \in \underset{x \in R_m^+ : \, p \cdot x \leq B_i}{\operatorname{argmax}} v_i(x)$$

- **Market clears:** For each good j,
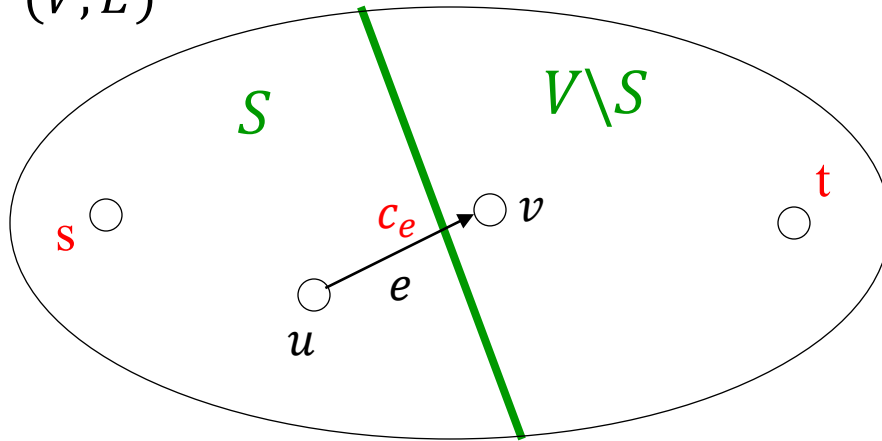$$\text{demand} = \text{supply}$$

Fairness and efficiency guarantees:
Pareto optimal (PO)
Weighted Envy-free
Weighted Proportional
Maximizes W. NW.

Algorithm: Set up as a "flow problem"

Max Flow (One slide overview)

Directed Graph
(V, E)

S $V \backslash S$

s c_e v t

e

u

Given $s, t \in V$. Capacity c_e for each edge $e \in E$.

Find maximum flow from s to t, $(f_e)_{e \in E}$ s.t.

- Capacity constraint
$$f_e \leq c_e, \ \forall e \in E$$

- Flow conservation: at every vertex $u \neq s, t$
total in-flow = total out-flow

Theorem: Max-flow = Min-cut
     s-t s-t

s-t cut: $S \subset V, \ s \in S, \ t \notin S$

cut-value: $C(S) = \sum_{\substack{(u,v) \in E: \\ u \in S, v \notin S}} c_{(u,v)}$

Min s-t cut: $\min_{\substack{S \subset V: \\ s \in S, t \notin S}} C(S)$

Can be solved in
strongly polynomial-time

CE Characterization

Pirces $p = (p_1, \ldots, p_m)$ and allocation $X = (x_1, \ldots, x_n)$

- **Optimal bundle:** Agent i demands $x_i \in \underset{x:\, p \cdot x \leq B_i}{\mathrm{argmax}}\, v_i(x)$

  - $p \cdot x_i = B_i$

  - $x_{ij} > 0 \Rightarrow \dfrac{v_{ij}}{p_j} = \underset{k \in G}{\max}\, \dfrac{v_{ik}}{p_k}$, for all good j

- **Market clears:** For each good j, demand $=$ supply

$$\sum_i x_{ij} = 1.$$

Competitive Equilibrium → Flow

Pirces $p = (p_1, \ldots, p_m)$ and allocation $F = (f_1, \ldots, f_n)$

$$f_{ij} = x_{ij} p_j \text{ (money spent)}$$

- **Optimal bundle:** Agent i demands $x_i \in argmax_{x:\ p \cdot x \leq B_i} v_i(x)$
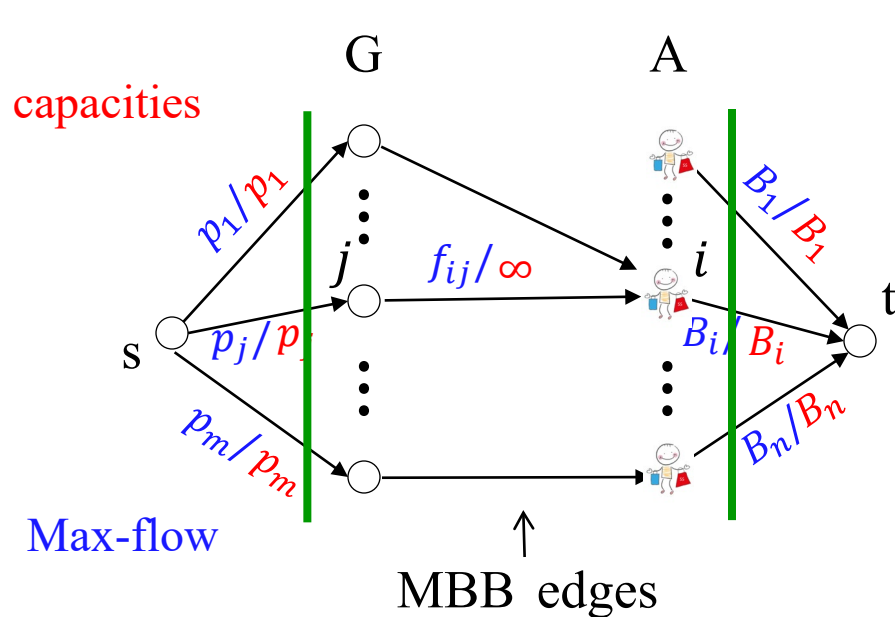
  - $\sum_{j \in G} f_{ij} = B_i$
  - $f_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \boxed{\max_{k \in G} \frac{v_{ik}}{p_k}}$ for all good j

    Maximum bang-per-buck (MBB)

- **Market clears:** For each good j, demand = supply

$$\sum_{i \in N} f_{ij} = p_j$$

Competitive Equilibrium → Flow

capacities

G   A

CE: (p, F) s.t.

Opt. Bundle $\begin{cases} \sum_{j \in M} f_{ij} = B_i \\ \\ f_{ij} > 0 \text{ on MBB edges} \end{cases}$

Market clears $\left\{ \sum_{i \in N} f_{ij} = p_j \right.$

Max-flow

MBB edges

Max-flow = min-cut
$= \sum_{j \in G} p_j = \sum_{i \in A} B_i$

Issue: Eq. prices and hence also MBB edges not known!

Fix [DPSV'08]: Start with low prices, keep increasing.

Maintain:

1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are fully sold)
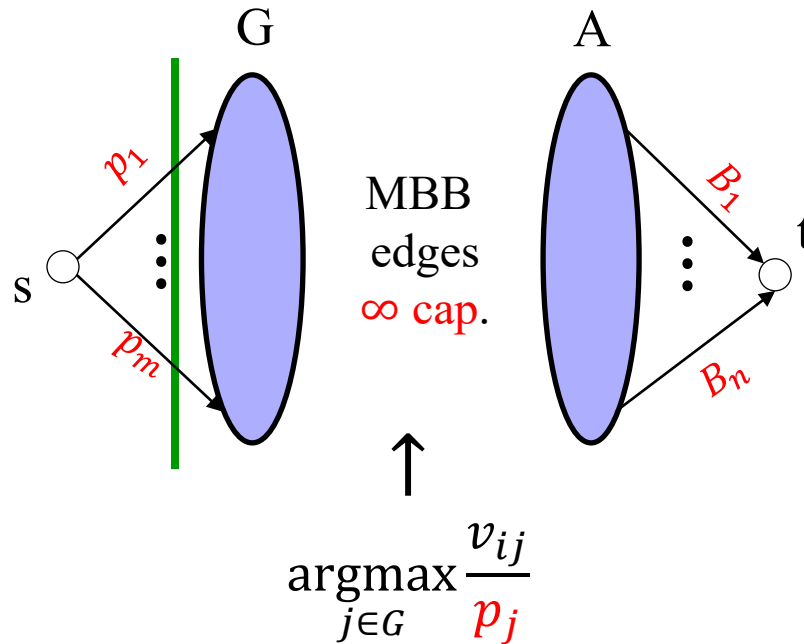
Algorithm (Pictorial)
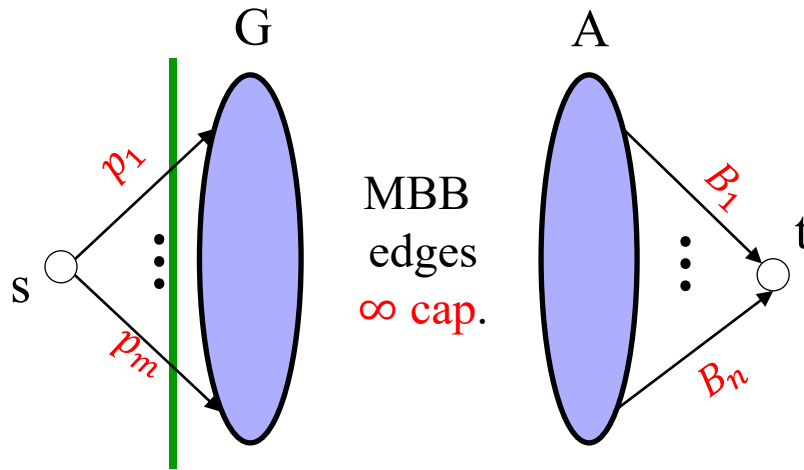
Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

G

A

p_1

p_m

MBB edges
∞ cap.

B_1

B_n

s

t

Init: $\forall j \in G$, $p_j < \min_i \dfrac{B_i}{m}$, and at least one MBB edge to j
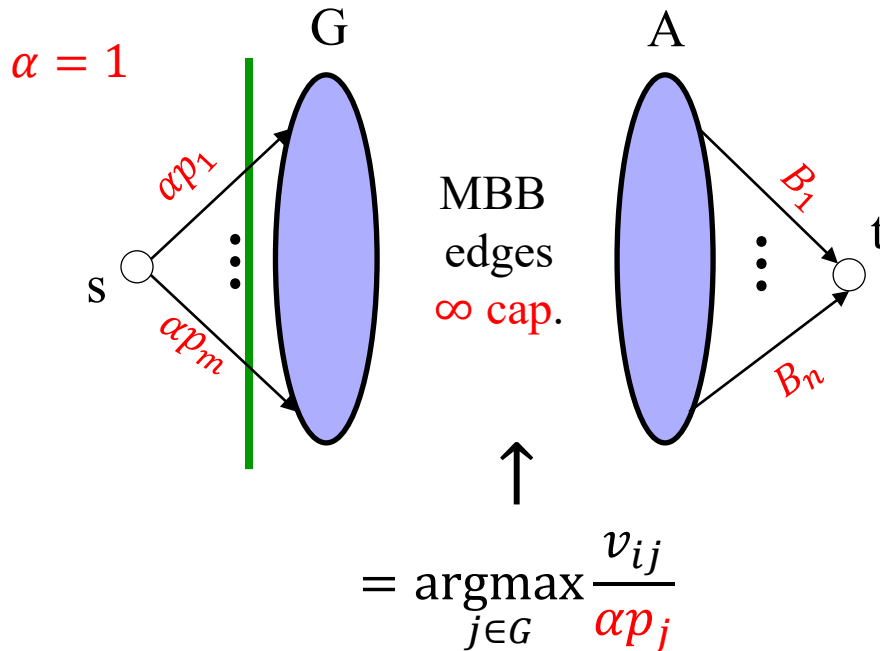
Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

G   A

p_1

p_m

s

t

MBB
edges
∞ cap.

B_1

B_n

$$\underset{j \in G}{\operatorname{argmax}} \frac{v_{ij}}{p_j}$$

Init: $\forall j \in G,\ p_j < \min_i \frac{B_i}{m}$, and
at least one MBB edge to j
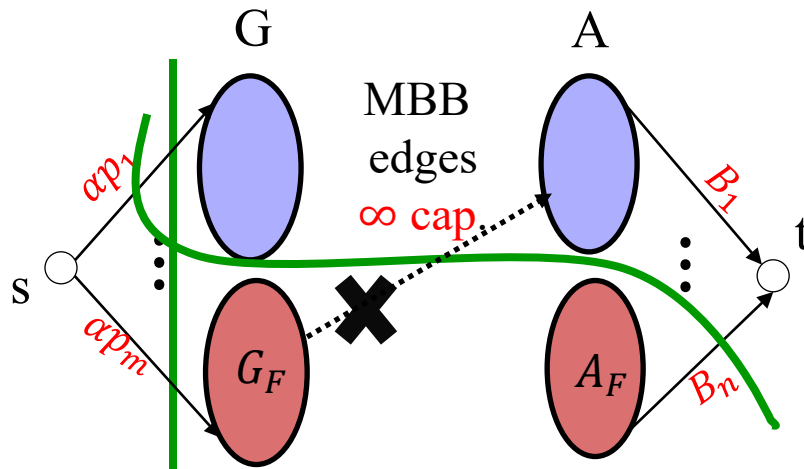
Increase p:

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

$\alpha = 1$

G   A

αp_1

αp_m

s

MBB edges ∞ cap.

B_1

B_n

t

$$= \underset{j \in G}{\mathrm{argmax}} \frac{v_{ij}}{\alpha p_j}$$

Init: $\forall j \in M, \ p_j < \min_i \frac{B_i}{n}$

And at least one MBB edge to j

Increase p: $\uparrow \alpha$

R. Mehta (ADFOCS'20)

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

Init: $\forall j \in M, \ p_j < \min_i \frac{B_i}{n}$

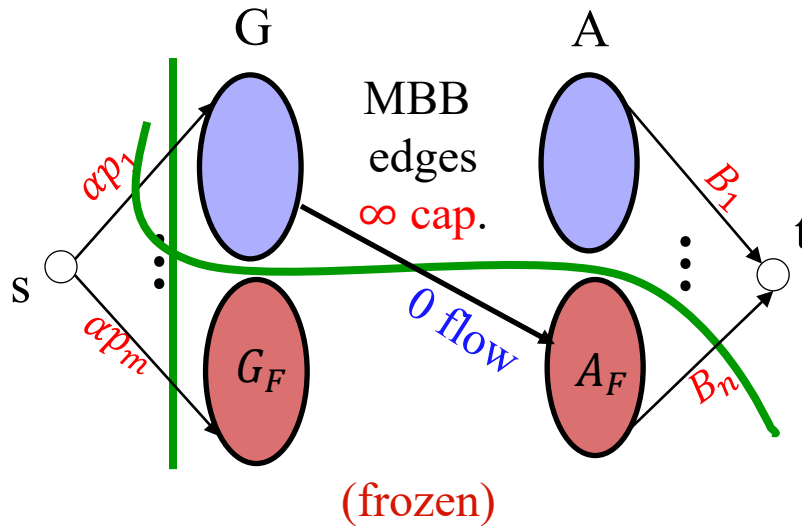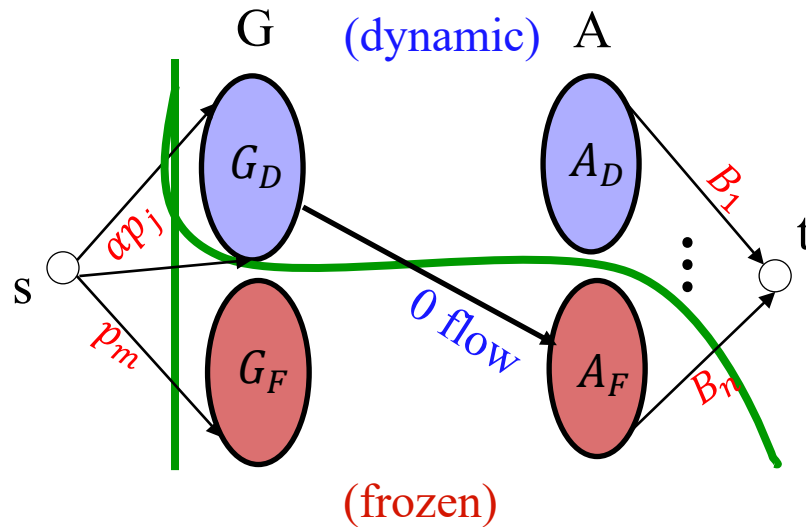And at least one MBB edge to j

Increase p: $\uparrow \alpha$

Event 1: New cross-cutting min-cut

Agents in A_F exhaust all their money.

G_F: Goods that have MBB edges only from A_F.

A **tight-set.**

Observation: If α is increased further, then G_F can not be fully sold. And $\{s\}$ will cease to be a min-cut.

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

G   A

MBB edges

∞ cap.

αp_1

αp_m

G_F

A_F

B_1

B_n

0 flow

s

t

(frozen)

Init: $\forall j \in M, \; p_j < \min_i \dfrac{B_i}{n}$

And at least one MBB edge to j

Increase p: $\uparrow \alpha$

Event 1: A tight subset G_F
Call it *frozen:* (G_F, A_F).

R. Mehta (ADFOCS'20)

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

G (dynamic) A

G_D A_D

αp_j

s

p_m

G_F A_F

0 flow

B_1

B_n

t

(frozen)

Init: $\forall j \in M, \ p_j < \min_i \frac{B_i}{n}$

And at least one MBB edge to j

Increase p: $\uparrow \alpha$

Event 1: A tight subset G_F

Call it *frozen:* (G_F, A_F).

Freeze prices in G_F.

Increase prices in G_D.

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

G   (dynamic)   A

G_D

A_D

S N(S)

s

αp_j

p_m

B_1

t

G_F

A_F

B_n

(frozen)

Observation: If α is increased further, then **S** can not be fully sold. And $\{s\}$ will cease to be a min-cut.

Init: $\forall j \in M,\ p_j < \min_i \dfrac{B_i}{n}$

And at least one MBB edge to j

Increase p: $\uparrow \alpha$

Event 1: A tight subset $S \subseteq G_D$

$N(S)$: *Neighbors of S*

Move $(S, N(S))$ *from dynamic to frozen.*

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

G   (dynamic)   A

G_D

A_D

B_1

s

αp_j

p_m

S

G_F

N(S)

A_F

B_n

t

(frozen)

Init: $\forall j \in M, \ p_j < \min\limits_{i} \dfrac{B_i}{n}$
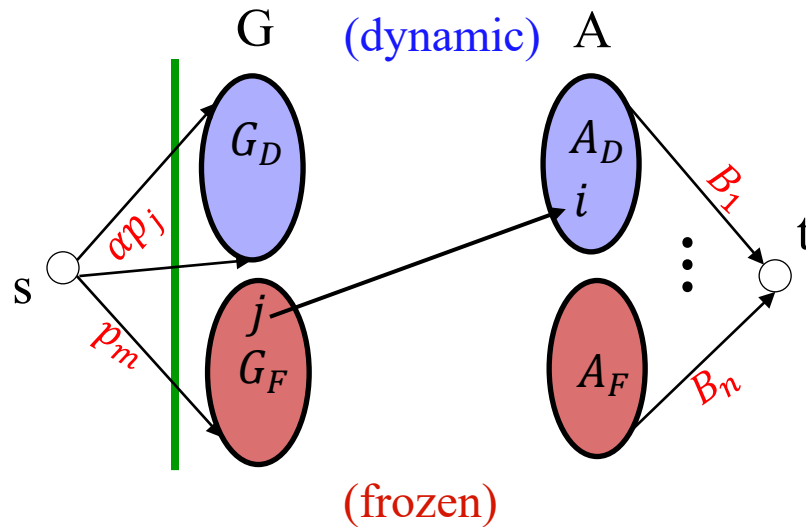
And at least one MBB edge to j

Increase p: $\uparrow \alpha$

Event 1: A tight subset $S \subseteq G_D$

Move $(S, N(S))$ to frozen part

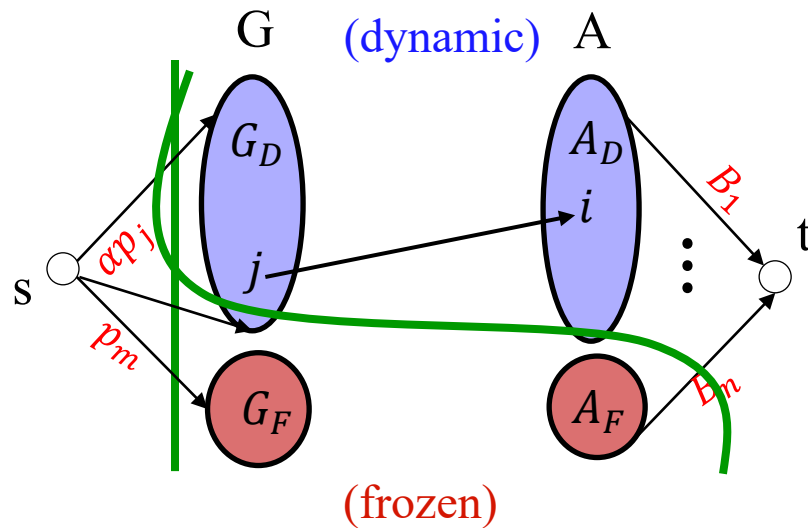*Freeze prices in G_F, and increase in G_D.*

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

G (dynamic) A

G_D

A_D
i

G_F
j

A_F

αp_j

p_m

B_1

B_n

s

t

(frozen)

Init: $\forall j \in M, \ p_j < \min_i \frac{B_i}{n}$

And at least one MBB edge to j

Increase p: $\uparrow \alpha$

Event 1: A tight subset $S \subseteq G_D$

Move $(S, \mathrm{N}(S))$ from dynamic to frozen
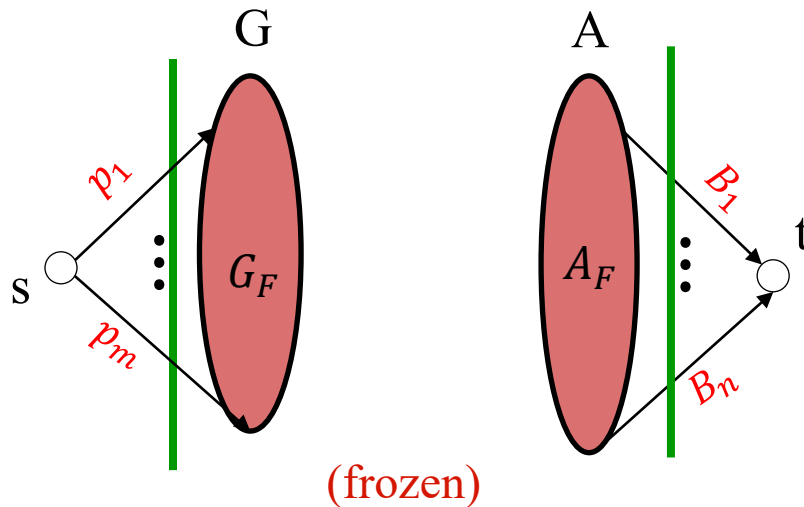
Freeze prices in G_F, and
increase in G_D.

OR

Event 2: New MBB edge

Must be between $i \in A_D$ & $j \in G_F$.
Recompute dynamic and frozen.

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut $= \{s\}$ (goods are sold)

G (dynamic) A

G_D A_D i B_1 t

αp_j j   s

p_m

G_F A_F B_n

(frozen)

Init: $\forall j \in M, \ p_j < \min_i \frac{B_i}{n}$

And at least one MBB edge to j

Increase p: $\uparrow \alpha$

Event 1: A tight subset $S \subseteq G_D$

Move $(S, \mathrm{N}(S))$ from dynamic to frozen
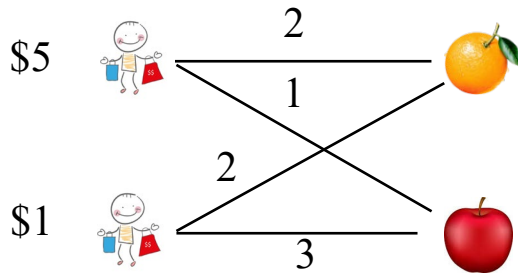Freeze prices in G_F, and
increase in G_D.

OR

Event 2: New MBB edge

Has to be from $i \in A_D$ to $j \in G_F$.
Recompute dynamic and frozen:
*Move the component containing
good j from frozen to dynamic.*

Algorithm (Pictorial)

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

G   A

p_1

G_F A_F

p_m

s B_1 t

B_n

(frozen)

Init: $\forall j \in M,\ p_j < \min_i \frac{B_i}{n}$
And at least one MBB edge to j

Increase p: $\uparrow \alpha$

Event 1: A tight subset $S \subseteq G_D$
Move $(S, \mathrm{N}(S))$ from dynamic to frozen
Freeze prices in G_F, and
increase in G_D.

OR

Event 2: New MBB edge
Must be from $i \in A_D$ to $j \in G_F$.
Recompute dynamic and frozen.

Stop: all goods are frozen.

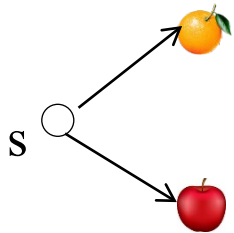Observations: Prices only increase.
Each increase can be lower bounded.
Both the events can be computed
efficiently.

⇩

Converges to CE in finite time.

Example

Invariants
1. Flow only on MBB edges
2. Min-cut = $\{s\}$ (goods are sold)

Input

$5

$1

2

1
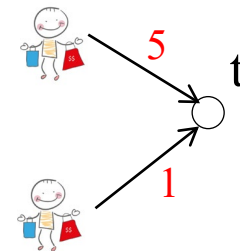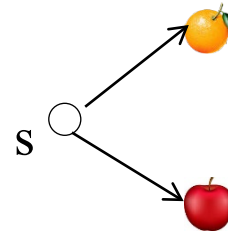
2

3

Init.

5

1

s

t

Event 1

5

1

s

t

Event 2

5

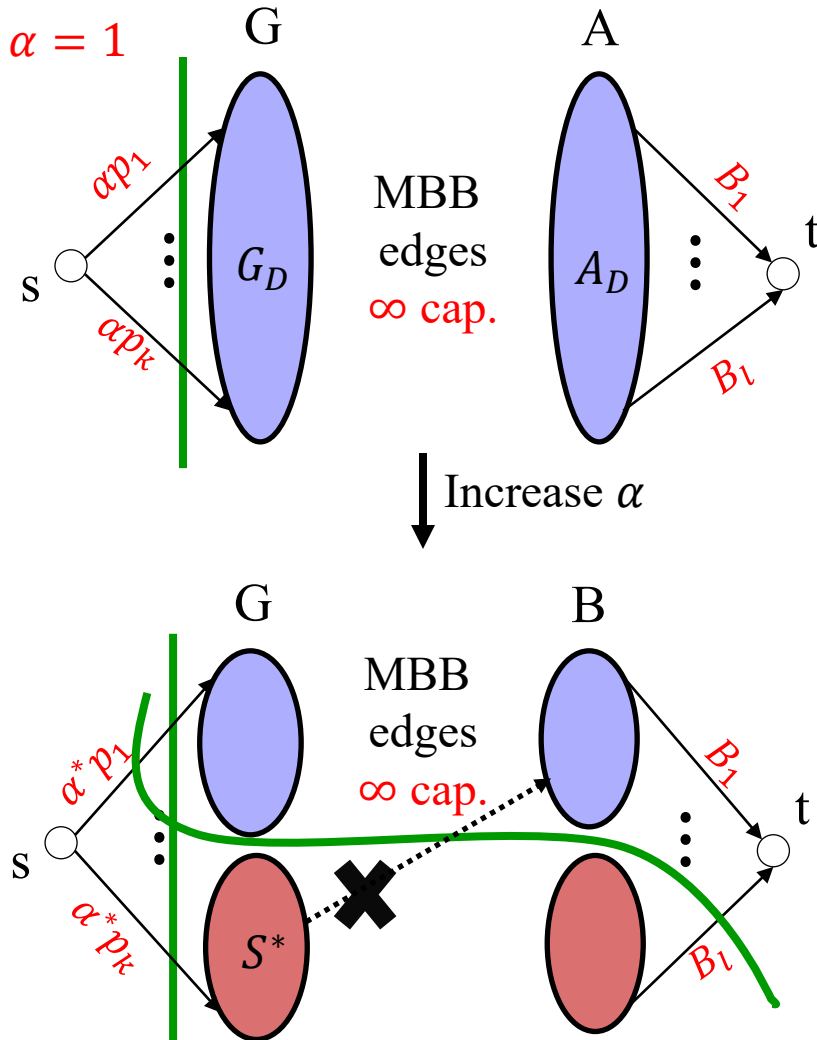1

s

t
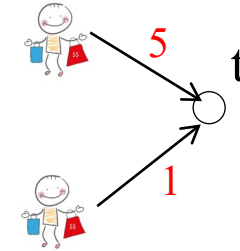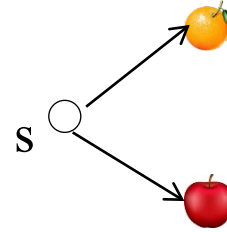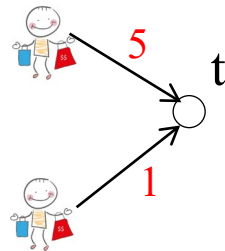
Formal Description

- Init: $p \leftarrow$ "low-values" s.t. $\{s\}$ is a min-cut. $(G_D, A_D) \leftarrow (G, A), \ (G_F, A_F) \leftarrow (\emptyset, \emptyset)$

- While($G_D \neq \emptyset$)

  - $\alpha \leftarrow 1, \ p_j \leftarrow \alpha p_j \ \forall j \in G_D$. Increase α until

    Event 1: Set $S \subseteq G_D$ becomes tight.

    $N(S) \leftarrow$ agents w/ MBB edges to S (neighbors of S).

    Move $(S, N(S))$ from (G_D, A_D) to (G_F, A_F).

    Event 2: New MBB edge appears between $i \in A_D$ and $j \in G_F$

    Add $(j \rightarrow i)$ edge to graph.

    Move component of j from (G_F, A_F) to (G_D, A_D).

- Output (p, F)

Efficiently Computing Event 2

Event 2: New MBB edge appears between $i \in A_D$ and $j \in G_F$

Exercise ☺

Efficiently Computing Event 1

$\alpha = 1$

G   A

αp_1

s

G_D

MBB edges
∞ cap.

A_D

t

B_1

αp_k

B_ι

Increase α

G   B

$\alpha^* p_1$

s

MBB edges
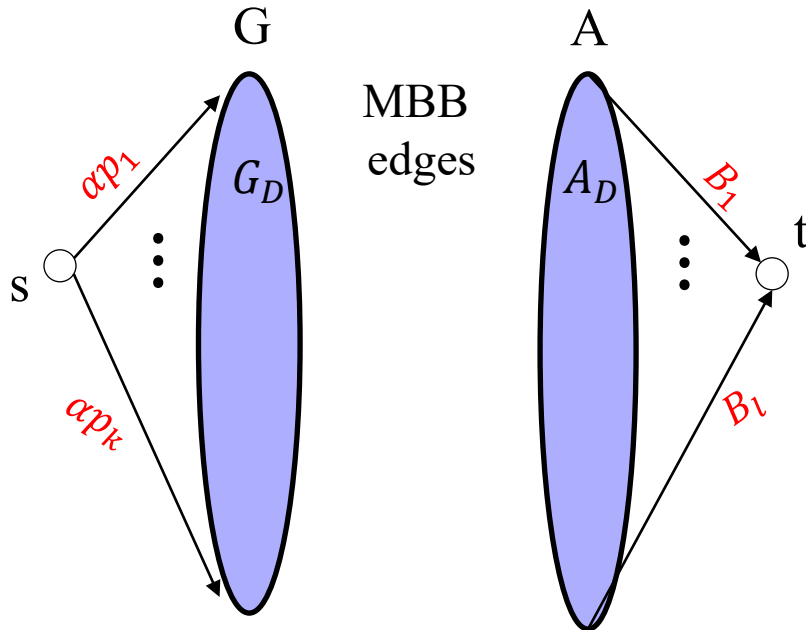∞ cap.

B_1

t

S^*

$\alpha^* p_k$

B_ι

Event 1: Set $S^* \subseteq G_D$ becomes tight.

- $\alpha^* = \dfrac{\sum_{i \in N(S^*)} B_i}{\sum_{j \in S^*} p_j}$

$$= \min_{S \subseteq G_D} \boxed{\dfrac{\sum_{i \in N(S)} B_i}{\sum_{j \in S} p_j}} \; \alpha(S)$$

- Find $S^* = \underset{S \subseteq G_D}{\operatorname{argmin}} \; \alpha(S)$
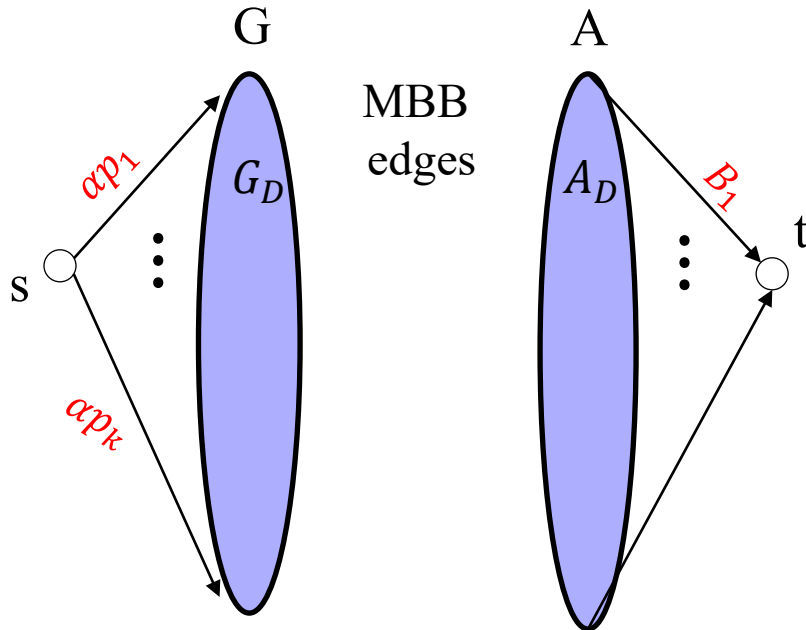
Efficiently Computing Event 1

Event 1: Set $S^* \subseteq G_D$ becomes tight.

- $\alpha^* = \dfrac{\sum_{i \in N(S^*)} B_i}{\sum_{j \in S^*} p_j}$

- $= \min\limits_{S \subseteq G_D} \boxed{\dfrac{\sum_{i \in N(S)} B_i}{\sum_{j \in S} p_j}} \rangle \, \alpha(S)$

- Find $S^* = \operatorname*{argmin}\limits_{S \subseteq G_D} \alpha(S)$

Efficiently Computing Event 1

G

A

MBB
edges

G_D

A_D

αp_1

αp_k

B_1

s

t

- $\alpha(S) = \frac{\sum_{i \in N(S)} B_i}{\sum_{j \in S} p_j}$

Find $S^* = \underset{S \subseteq G_D}{\mathrm{argmin}}\ \alpha(S)$

Claim. Can be done in O(n) min-cut computations

$(G', A') \leftarrow (G_D, A_D)$

Repeat{

$\quad \alpha \leftarrow \alpha(G')$. Set $c_{(s,j)} \leftarrow \alpha p_j, \forall j \in G'$

$\quad (s \cup \{S\} \cup N(S)) \leftarrow$ min-cut in (G', A')

$\quad (G', A') \leftarrow (S, N(S))$

}Until($\{s\}$ not a min-cut)

Return α

Efficient Flow-based Algorithms

- **Polynomial running-time**
  - Compute *balanced-flow:* minimizing l_2 norm of agents' surplus [DPSV'08]
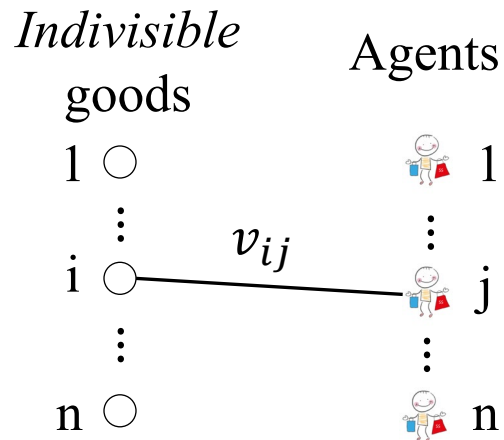- **Strongly polynomial: Flow + scaling** [Orlin'10]

Exchange model (barter):

- **Polynomial time** [DM'16, DGM'17, CM'18]
- **Strongly polynomial for exchange**
  - Flow + scaling + approximate LP [GV'19]

Hylland-Zeckhauser
(an extension)

Motivation: Matching

Indivisible goods

Agents

1 ○

i ○ v_{ij} ─── j

n ○ n

Goal: Design a method to match goods to agents so that
- The outcome is **Pareto-optimal** and **envy-free**
- **Strategy-proof**: Agents have no incentive to lie about their $v_{ij}s$.

Hylland-Zeckhauzer'79: Compute CEEI where every agent wants total amount of at most one unit.

But the outcome is a fractional allocation!

Think of it as probabilities/time-shares/… []

HZ Equilibrium

Given:

- Agents $A = \{1, \ldots, n\}$, indivisible goods $G = \{1, \ldots, n\}$
- v_{ij}: value of agent i for good j.
  - ☐ If i gets j w/ prob. x_{ij}, then the expected value is: $\sum_{j \in G} v_{ij} x_{ij}$

Want: prices $p = (p_1, \ldots, p_n)$, allocation $X = (x_1, \ldots, x_n)$

- Each good j is allocated: $\sum_{i \in A} x_{ij} = 1$
- Each agent i gets an optimal bundle subject to
  - ☐ $1 budget, and **unit allocation**.

$$x_i \in \operatorname*{argmax}_{x \in R_+^m} \left\{ \sum_j v_{ij} x_j \,\middle|\, \sum_j x_j = 1, \sum_j p_j x_j \leq 1 \right\}$$

HZ Equilibrium

Hyllander-Zeckhauzer'79

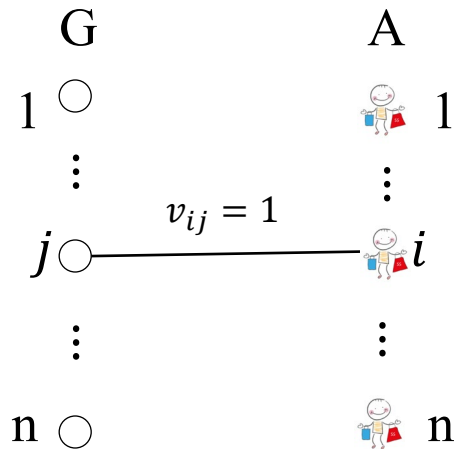- Exists. Pareto optimal, Strategy proof in large markets.

Vazirani-Yannakakis'20

- Irrational equilibrium prices $\Rightarrow$ not in PPAD

- In FIXP

- Algorithm for bi-valued preferences:

$$v_{ij} \in \{a_i, b_i\} \text{ where } a_i, b_i \geq 0$$

VY'20 Algorithm
$(v_{ij} \in \{0,1\})$

G   A

1 ○   1

⋮   ⋮

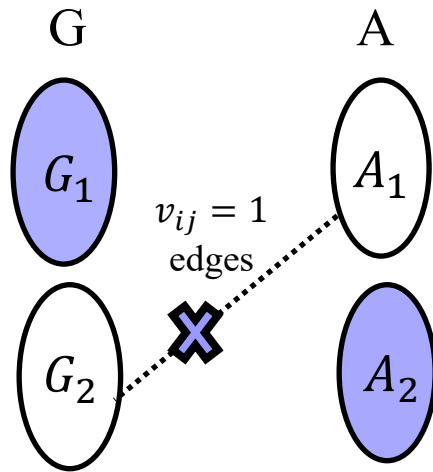$v_{ij} = 1$

j ○————————— i

⋮   ⋮

n ○   n

At equilibrium, an agent's utility is at most 1.

Perfect matching $\Rightarrow$ An equilibrium is,

- Allocation on the matching edges
- Zero prices

VY'20 Algorithm
$(v_{ij} \in \{0,1\})$

G   A

G_1 A_1

$v_{ij} = 1$
edges

G_2 A_2

No perfect matching

- Min vertex cover: $(G_1 \cup A_2)$
  - No $A_1 - G_2$ edge

VY'20 Algorithm
$(v_{ij} \in \{0,1\})$

G   A

G_1

$N(S)$ $v_{ij} = 1$ edges

G_2

A_1

S

A_2

No perfect matching

- Min vertex cover: $(G_1 \cup A_2)$
  - ☐ No $A_1 - G_2$ edge
  - ☐ For each $S \subseteq A_2$, $|N(S) \cap G_2| \geq |S|$
    - Else get smaller VC by replacing S with $N(S) \cap G_2$
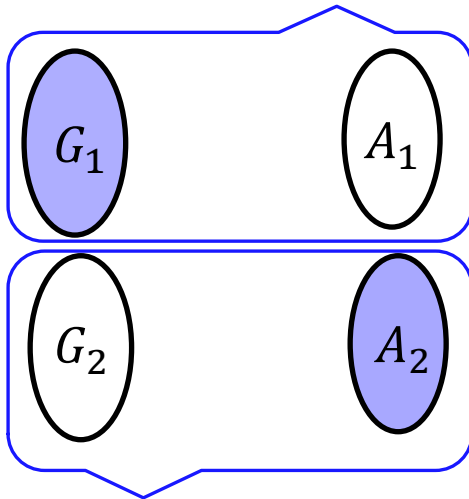
Max matching in (G_2, A_2) matches all of A_2. ⬅ Subgraph (G_2, A_2) satisfies hall's condition for A_2.

VY'20 Algorithm
$(v_{ij} \in \{0,1\})$

CEEI

Max matching

No perfect matching

- Min vertex cover: $(G_1 \cup A_2)$
  - □ No $A_1 - G_2$ edge
  - □ For each $S \subseteq A_2, |N(S) \cap G_2| \geq |S|$
    - Max matching in (G_2, A_2) matches all of A_2.

VY'20 Algorithm
$(v_{ij} \in \{0,1\})$

Prices

p at CEEI

CEEI

G_1 A_1

0
⋮
0

G_2 ✖ A_2

Max matching

Running-time:
Strongly polynomial

Want: (p, X)
Each good j is sold (1 unit)
Each agent i gets
$$x_i \in \underset{x:\sum_j x_j = 1, \sum_j p_j x_j \leq 1}{\mathrm{argmax}} \sum_{j \in G} v_{ij} x_j$$

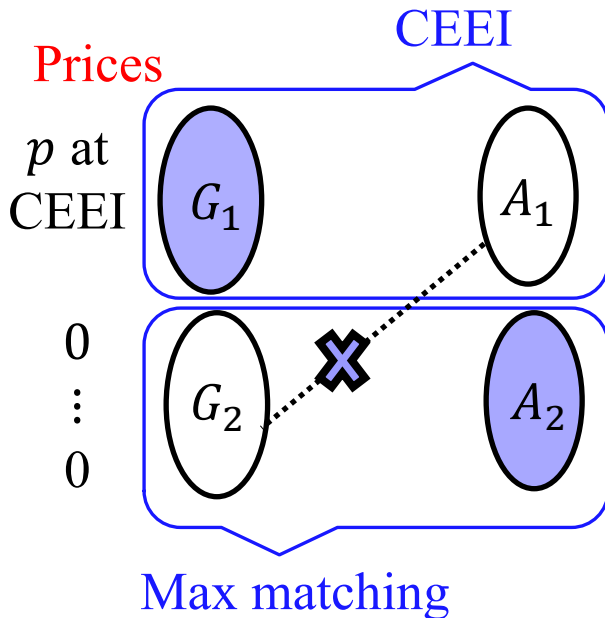## No perfect matching

■ Min vertex cover: $(G_1 \cup A_2)$

■ Eq. Prices: CEEI prices for G_1, and
    0 prices for G_2

■ Eq. Allocation
  □ $i \in A_2$ gets her matched good 😊
  □ $i \in A_1$ gets CEEI allocation + unmatched goods from G_2 😊

VY'20 Algorithm

bi-values: $v_{ij} \in \{a_i, b_i\}, 0 \leq a_i < b_i$

Reduces to $v_{ij} \in \{0,1\}$

Exercise.

Open Questions

HZ Equilibrium

Computation for the general case.

Is it hard? OR is it (approximation) polynomial-time?

- Efficient algorithm when #goods or #agents is a constant [DK'08, AKT'17]
    - Cell-decomposition and enumeration

What about chores?

- CEEI exists but may form a non-convex set [BMSY'17]

- Efficient Computation?
  - ☐ **Open: Fisher as well as for CEEI**
  - ☐ For constantly many agents (or chores) [BS'19, GM'20]
  - ☐ *Fast* path-following algorithm [CGMM.'20]

- Hardness result for an exchange model [CGMM.'20]

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THANK YOU