LECTURE 20 (April 1st)

TODAY Quantum PCP Conjecture

RECAP Motivating question

"Which local Hamiltonians have ground states (or low-energy states) that are simple?"
↳ computable by polynomial depth circuits

Classical PCP Theorem

The PCP Theorem gives a robust version of the statement that SAT is NP-hard:

PCP Theorem $\forall \varepsilon > 0$ and any 3SAT instance φ, deciding if $\mathrm{MAXSAT}(\varphi) = 1$ or $\mathrm{MAXSAT}(\varphi) \leq 7/8 + \varepsilon$ is NP-hard

So, the problem goes from NP-hard to trivial and even approximating it to a factor $7/8$ is hard

If we encode 3SAT as a 3-Local Hamitonian Problem

$$H = \frac{1}{m} \sum_{i=1}^{m} H_i \quad \text{where} \quad \langle x | H_i | x \rangle = \begin{cases} 1 & \text{if } x \text{ violates } i^{th} \text{ clause} \\ 0 & \text{o/w} \end{cases}$$

Then, PCP theorem says that deciding if

$$\lambda_{min}(H) = 0 \quad \text{or} \quad \lambda_{min}(H) \geq \frac{1}{8} - \varepsilon \quad \text{is NP-hard}$$

As the name suggests, the proof relies on the idea of a probabilistically checkable proof

Def Let $L \in NP$. We say L has a probabilistically checkable proof if $\exists$ randomized poly-time verifier that queries $O(1)$ bits of the proof s.t.

(1) $x \in L \implies \exists$ proof π s.t. $\mathbb{P}[V \text{ accepts } (x,\pi)] \geq 2/3$
(2) $x \notin L \implies \forall$ proofs π $\quad \mathbb{P}[V \text{ accepts } (x,\pi)] \leq 1/3$

A PCP is a proof that can be spot-checked. By reading a constant number of bits we can verify its correctness with confidence

The proof-checking formulation of the PCP theorem is then the statement

"every language $L \in NP$ has a probabilistically checkable proof"

This is one of the major breakthroughs in complexity and the proof is remarkable

We will not be able to cover it here but the basic idea is the following

For a languge like 3SAT, the PCP proof consists of encoding a satisfying assignment using a carefully designed error-correcting code that enables easy verification

To translate this statement back to the MAXSAT approximation, one must convert the checks performed by a PCP verifier into a 3SAT formula, using similar ideas to the Cook-Levin theorem which encodes the computational history of the verifier into a 3SAT formula

Quantum PCP Conjecture

The quantum PCP conjecture is similar where replace NP with QMA and 3SAT with K-Local Hamiltonian problem

Quantum PCP Conjecture

$\exists$ a family of k-Local Hamiltonians, one for each qubit size n

$$H = \frac{1}{m} \sum_{i=1}^{m} H_i \quad \text{Where } m = \mathrm{poly}(n)$$

such that deciding if $\lambda_{min}(H) \leq \alpha$ or $\lambda_{min}(H) \geq \beta$ is QMA-hard for universal constants k, α, β s.t. $\beta - \alpha = \Omega(1)$.

Note Energy gap is a constant here as opposed to inverse polynomial and also note that energy of any state $|\psi\rangle$ is always between $[0,1]$ by normalization

Consequences

If QMA = NP, then quantum PCP conjecture is true by the classical PCP theorem [Why?] so to have non-trivial results let us assume that QMA $\neq$ NP

Let us further assume, as is believed that QMA $\neq$ QCMA, i.e. quantum proofs are more powerful than classical proofs

Then, the QPCP conjecture has some profound consequences about complexity and entanglement of ground states of local Hamiltonians

1 Classical Description of Low-energy States

We saw that the Local Hamiltonian problem with $1/\mathrm{poly}(n)$-gap is QMA-complete

If we are further guaranteed that in the ACCEPT case the witness state $|\psi\rangle$ has polynomial circuit depth, then this problem turns out to be QCMA-complete

So, one can think of QCMA as capturing the problem of trying to find a local Hamitonian's ground state with polynomial depth

The QPCP conjecture together with QMA $\neq$ QCMA implies that there are local Hamiltonians where not only ground states but all states of energy atmost β have super-polynomial circuit depth

2 Room temperature entanglement

For a Local Hamiltonian H, consider the mixed state

$$\rho_H(T) = \frac{e^{-H/T}}{\mathrm{Tr}(e^{-H/T})}$$

where $T \in [0, \infty)$ is the temperature

This is called the Gibbs state

e^A is the matrix exponential. If $A = \sum_i \lambda_i |u_i\rangle\langle u_i|$ is the spectral decomposition of A then

$$e^A = \sum_i e^{\lambda_i} |u_i\rangle\langle u_i|$$

It turns out (and this will be an optional homework exercise) that

Absolute zero	High temperature
$T \to 0$	$T \to \infty$
$\rho_H(T) =$ uniform mixture over ground states of H	$\rho_H(T) = \frac{I}{2^n}$ = maximally mixed state
Depth = superpoly(n) assuming QMA $\neq$ QCMA	Depth O(1)
Complex Entanglement	No entanglement

We know several physical phenomena such as superfluidity where complex entanglement is present near absolute zero temperature

At what temperature do we transition to complex entanglement?

QPCP conjecture says that complex entanglement can be present at "room temperature" and it is QMA-hard to compute the energy of the system even at such temperatures

Caveat Physically Relevant Hamiltonians have more structure, so it is still possible that Hamiltonians satisfying QPCP conjecture (if they exist) are not physically relevant

Evidence against QPCP

We know that 2-local Hamiltonian problem with $1/\text{poly}(n)$-gap is QMA hard

Note that a 2-Local term only acts on 2 qubits

It turns out that even if the qubits are arranged on a $\sqrt{n} \times \sqrt{n}$ grid and we only have Hamiltonian terms that act on neighbouring qubits, the problem still remains QMA-hard

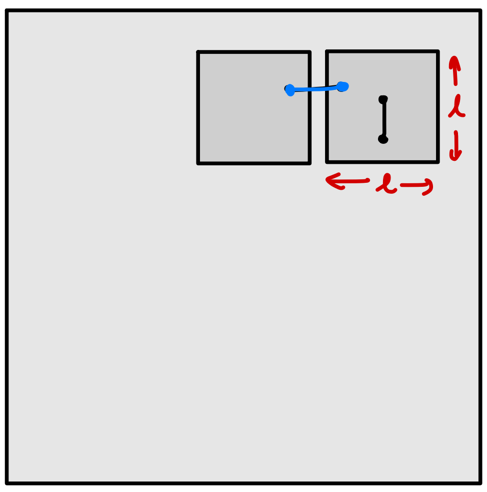
Assuming NP $\neq$ QMA (otherwise classical PCP theorem implies that QPCP is true) we claim that such grid Hamiltonians cannot be candidates for QPCP conjecture

$\boxed{\text{Claim}}$ For 2-Local Hamiltonians on a 2D-grid, $\exists\, |\psi\rangle$ with relatively high energy that has a small classical description

Let $H = \frac{1}{2n} \sum_{ij} H_{ij}$ be the 2-Local Hamiltonian on the grid

since there are 2n local terms

Divide the $\sqrt{n} \times \sqrt{n}$ grid into bunch of $\ell \times \ell$ square patches
Group all the local Hamiltonian terms that act on two qubits within a patch to make a "super-term". ℓ would be some constant.



We can write

$$H = \frac{1}{2n}\Big[\overbrace{\Big(\sum_{ij \in \text{patch}_1} H_{ij}\Big)}^{H'_1} + \overbrace{\Big(\sum_{ij \in \text{patch}_2} H_{ij}\Big)}^{H'_2} + \cdots$$

$$+ \cdots + \Big(\sum_{ij \in \text{patch}_T} H_{ij}\Big) + H_{\text{boundary}} \Big]$$

$H_{\text{boundary}} \to$ qubits in different patches

where $T = \frac{n}{\ell^2}$

How many boundary terms are there? At most $4\ell \cdot T = \frac{4n}{\ell}$

Each super term H_i' has a ground state on ℓ^2 qubits and the ground state $|\psi_i\rangle$ can be found in time $2^{O(\ell^2)}$ by brute force

Consider the state $|\psi\rangle = \bigotimes_{i=1}^{n/\ell^2} |\psi_i\rangle$ by tensoring the ground state of all patches

This is a state on n qubits but it has a short classical description and small depth

Description size $= O\left(\frac{n}{\ell^2} \cdot 2^{O(\ell^2)}\right) = O(n)$

& depth $= 2^{O(\ell^2)} = O(1)$ since $\ell = O(1)$

Let's compute the energy of $|\psi\rangle$:

$$\langle\psi|H|\psi\rangle = \frac{1}{2n}\left[\sum_{i=1}^{n/\ell^2} \underbrace{\langle\psi|H_i'\otimes \mathbb{I}|\psi\rangle}_{=\langle\psi_i|H_i'|\psi_i\rangle} + \underbrace{\langle\psi|H_{boundary}|\psi\rangle}_{\leq \frac{4n}{\ell}}\right]$$

since each local term $0 \preceq H_{ij} \preceq I$

$$\leq \frac{1}{2n}\sum_{i=1}^{n/\ell^2}\langle\psi_i|H_i'|\psi_i\rangle + \frac{2}{\ell}$$

On the other hand, suppose $|\theta\rangle$ be a ground state of H:

Then $\lambda_{min}(H) = \langle\theta|H|\theta\rangle$

$$\geq \frac{1}{2n}\sum_{i=1}^{n/\ell^2}\langle\theta|H_i'\otimes\mathbb{I}|\theta\rangle \quad \text{[dropping the boundary term]}$$

$$\geq \frac{1}{2n}\sum_{i=1}^{n/\ell^2}\langle\psi_i|H_i'|\psi_i\rangle \quad \text{[since } |\psi_i\rangle \text{ is the ground state of } H_i'\text{]}$$

Thus, energy of the state $|\psi\rangle$ above satisfies

$$\langle\psi|H|\psi\rangle \leq \lambda_{min}(H) + \frac{2}{\ell}$$

If we choose ℓ to be a large enough constant so that $\frac{2}{\ell} \ll \beta-\alpha$

Thus, we have constructed a state with poly(n)-sized classical description, constant-depth which approximates the ground energy to any constant precision

This wouldn't have been possible assuming QPCP conjecture

There is nothing special about 2D-grids here. Same argument work for any k-dimensional grid.

What the above suggests is that to prove QPCP conjecture one would need Hamiltonian terms interacting on a graph that cannot be chopped into patches, for example, graphs that look kind of random

An example of such graphs are expander graphs

Such graphs are very connected – if you take any set of vertices, then it has a lot of edges going out of it – and one can't decompose the graph into small patches where few edges go across patches

In fact, hard instances for the classical PCP theorem are based on expander graphs

Note: Although 2SAT is easy, one can define other constraint satisfaction problems on a graph that are instances of hard classical Hamiltonians

Based on this, one can wonder if we can have a 2-Local Hamiltonian on an expander graph that could prove QPCP conjecture

For classical PCPs, the better the expansion of the graph, the harder instances one gets

Surprisingly, Brandao and Harrow showed that this is not the case for QPCP

$\boxed{\text{Theorem}}$ Let H be a 2-local Hamiltonian on a graph G on n vertices with expansion $\geq \frac{1}{2} - \varepsilon$

i.e. # edges going out of a set $S \geq \left(\frac{1}{2} - \varepsilon\right)|S| \quad \forall\, |S| \leq n/4$

Then, $\exists$ a product state $|\phi_1\rangle \otimes \cdots \otimes |\phi_{n/t}\rangle$ where each $|\phi_i\rangle$ is on t-qubits such that

$$\langle \phi | H | \phi \rangle \leq \lambda_{min}(H) + \varepsilon^{1/3}$$

Thus, extremely good expanders cannot be QPCP candidates

There is a similar result for graphs with very high degree: if graph has degree D, then

$\exists$ a product state $|\phi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$ s.t.

$$\langle \phi | H | \phi \rangle \leq \lambda_{min} + D^{-1/6}$$

So, as D increases, one can find better and better approximations

This sort of behavior does not happen classically – more edges mean more constraints to satisfy

However, in the quantum setting, more constraints could force the problem to become more classical and hence easier

What this tells us is that to prove QPCP conjecture, we have to look for a "Goldilocks" family of Local Hamiltonians – not too sparse or dense, not too much like a grid or an expander, and so on

Evidence for QPCP

Recall that one consequence of QPCP together with QMA $\neq$ QCMA is that there are family of Local Hamiltonians s.t. all states $|\psi\rangle$ with energy atmost a constant β require circuit of superpolynomial depth (for instance, they are far from being a product state)

Infact, QPCP even implies stronger consequences but let's stick with the above

The No Low-energy Trivial States (NLTS) conjecture was a weaker version of this statement and was recently proven

Theorem (NLTS) $\exists$ family of Local Hamiltonians s.t. every state of energy at most some constant ε, require circuits of super-constant depth

These are non-trivial in the sense that they are not product states

The Hamiltonian family here is based on very good quantum LDPC error-correcting codes that were recently shown to exist

At the moment, it seems that the next step towards QPCP will require entirely new ideas

NEXT TIME Tensor Networks & Area Laws