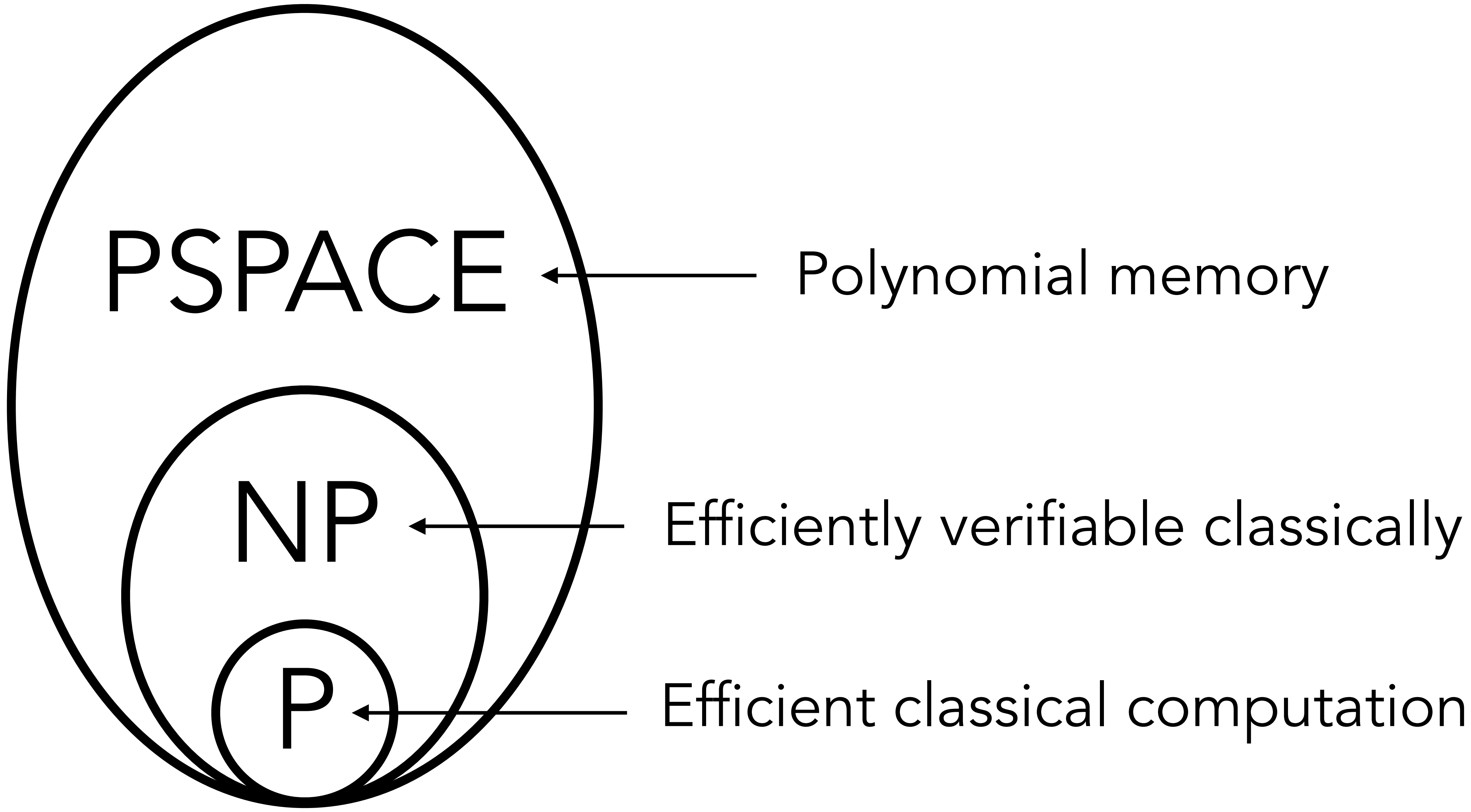
Computational Complexity Basics

A very short introduction

Complexity Theory

Study of computational resources needed
to solve different problems

Time? Memory?
Randomness?
Proofs? Interaction?



Turing machine and Decision Problems

Turing Machine mathematically formalizes what an algorithm is

You can think of it as a piece of code in some programming language that takes an input and gives an output

Turing Machine may take some auxiliary inputs such as random bits, advice, etc.

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Decision Problems

Typically, we are interested in resources (e.g. time, space) required for computing functions

$$f : \{0,1\}^* \to \{0,1\}$$

Sometimes promise functions or promised languages or search problems are considered

Languages, problems, functions are often used synonymously

Or equivalently, deciding whether a bitstring is in a language

A language is a subset of $\{0,1\}^*$
e.g. $L = \{x \mid f(x) = 1\}$

The Landscape of Complexity

The Landscape of Complexity

called **undecidable problems**

There are problems that cannot be solved by a Turing machine in any finite time

e.g. HALT(c, x)=1 iff the Turing machine whose code is c halts in finite time on input x

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P is the class of languages which can be decided with a Turing Machine that runs in polynomial time in the input length

Typically, we call polynomial run time as efficient

e.g. Linear Programming = Decide if a system of linear inequalities $a_i^\top x \leq b_i$ has a solution

e.g. 2SAT = Given a set of boolean 2-clauses (e.g. $x_i \vee \overline{x_j}$) decide if there is an assignment satisfying all clauses

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NP is the class of languages which can be efficiently decided when the Turing machine is given a polynomial-sized witness

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P $\subseteq$ NP but whether it is a strict subset is a Millennium Prize Problem

Why?

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It is believed that randomness does not help efficiency in computation, i.e. BPP=P

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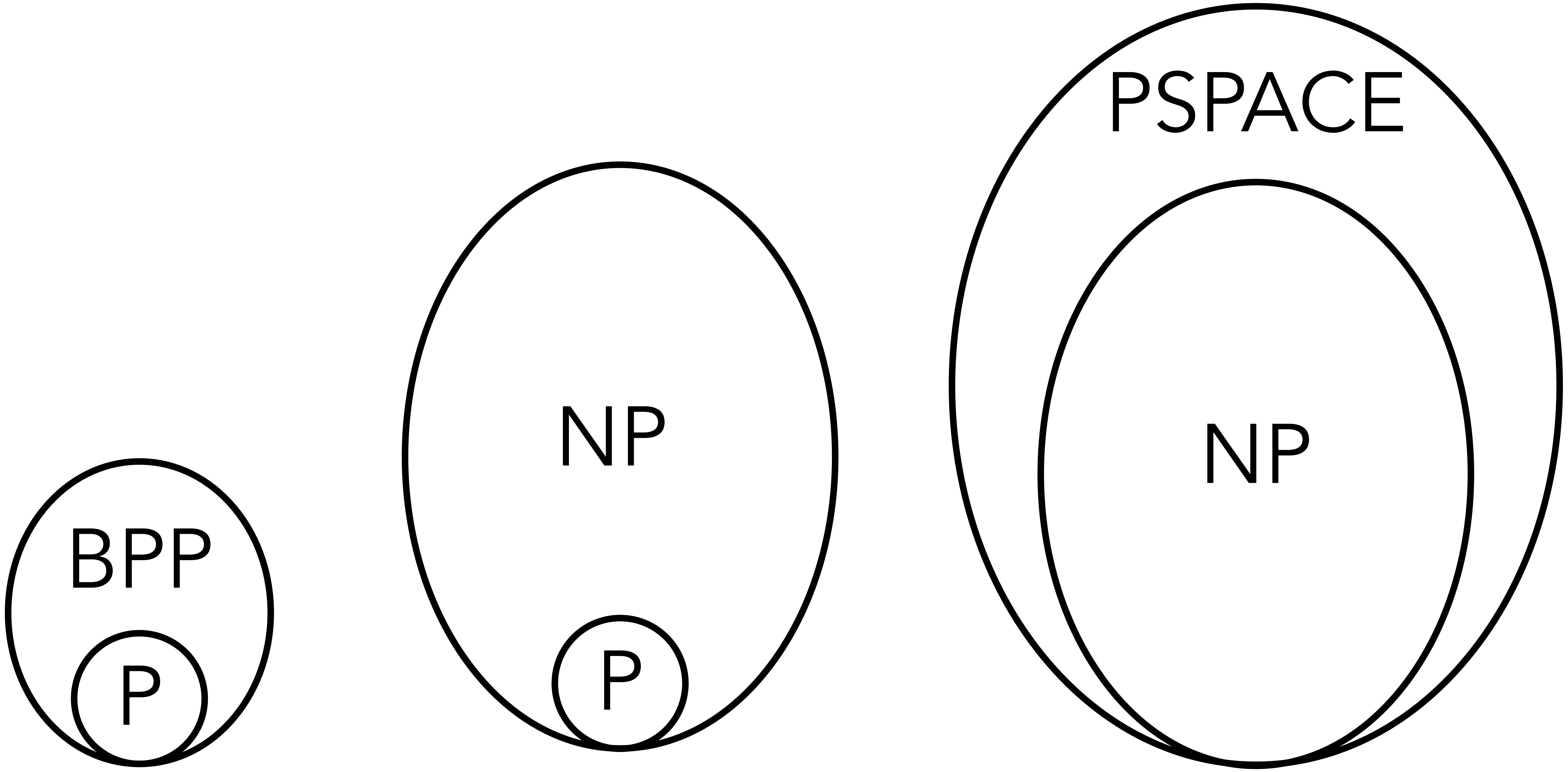

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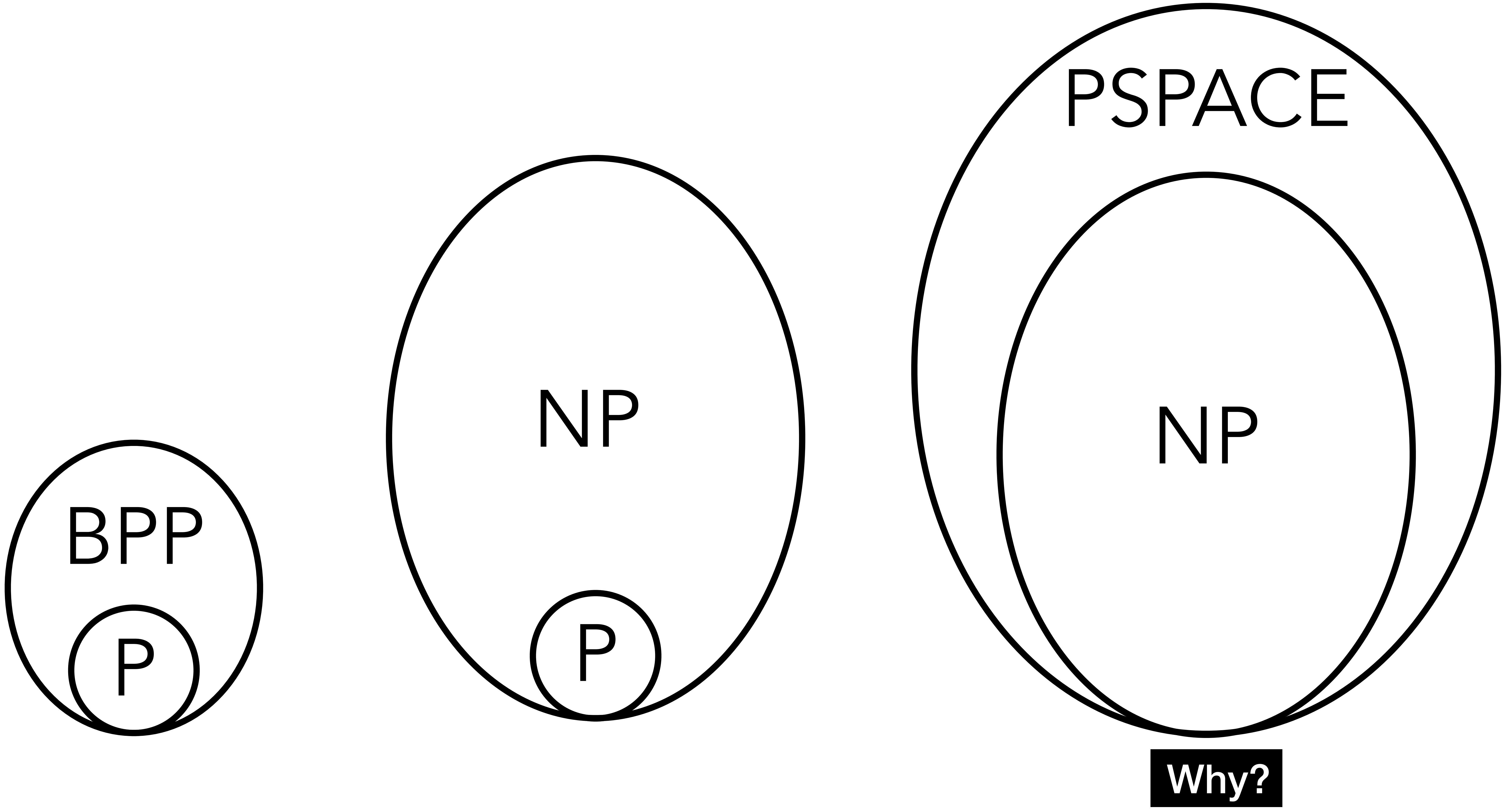

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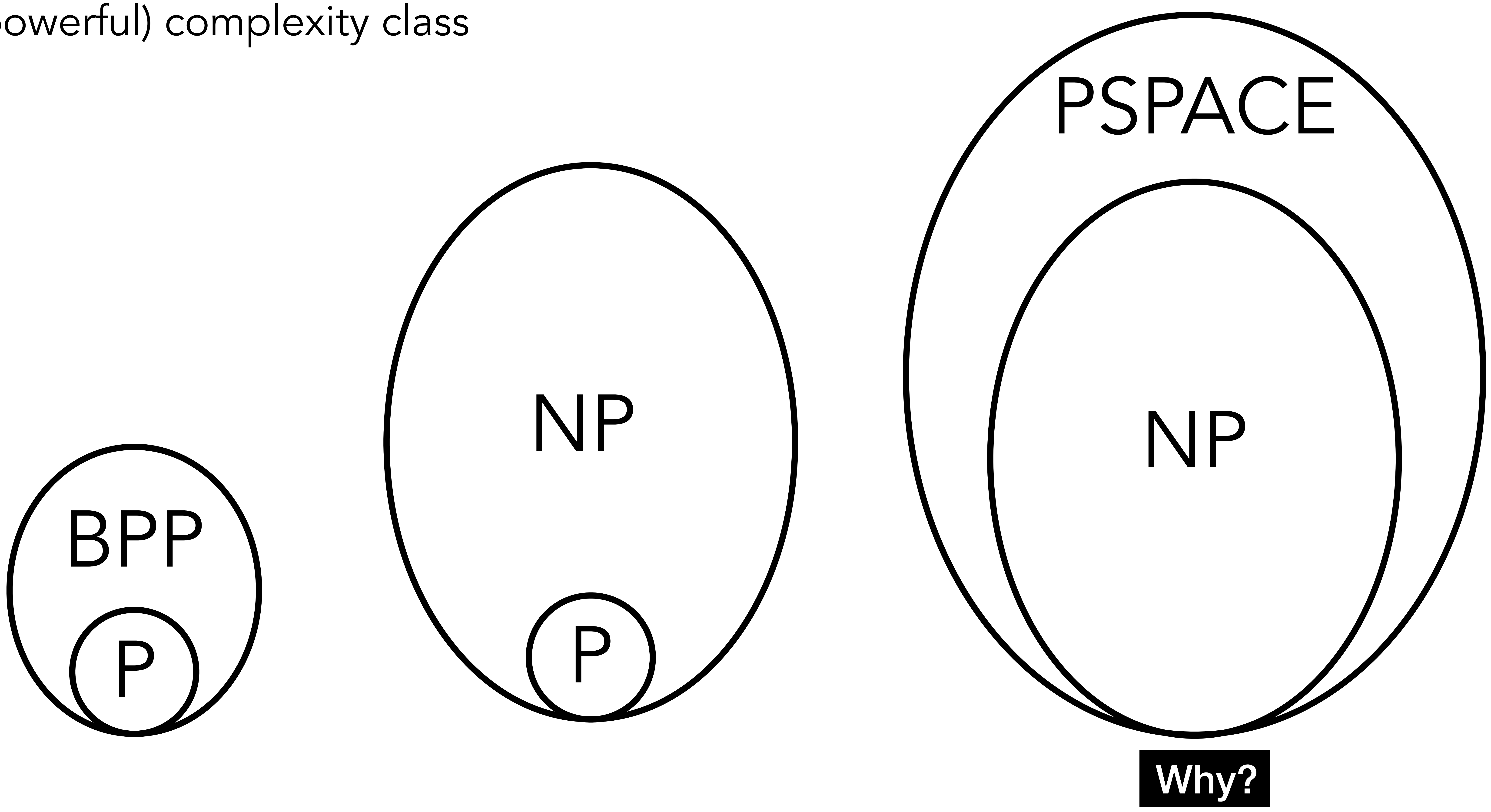

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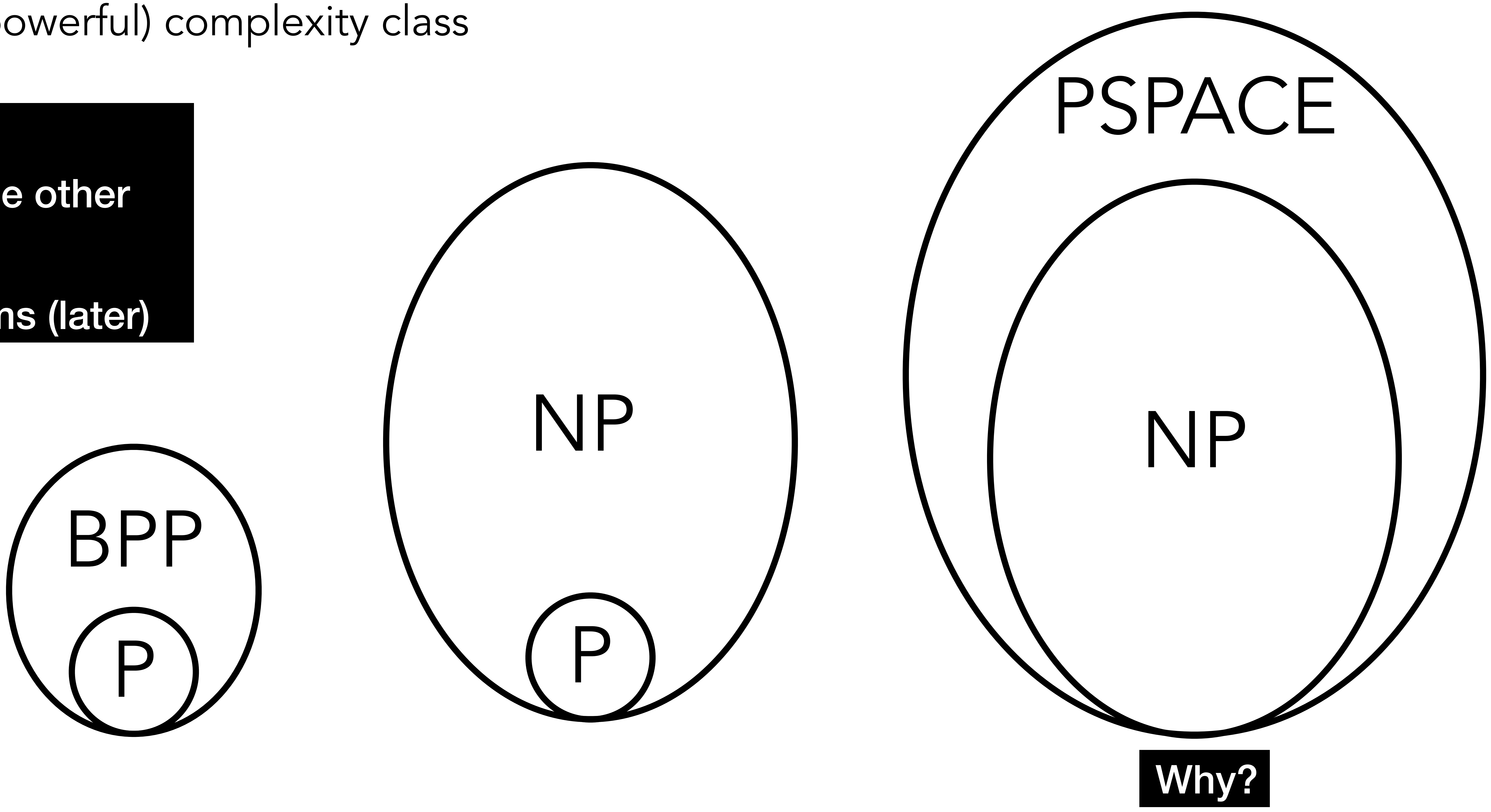
Alternatives?

- Say one problem is harder than the other
- Use Oracles (in a later lecture)
- Study restricted types of algorithms (later)



Reductions and Complete Problems

Reduction This formalizes that one problem is harder than another

$L_{easy} \leq L$ if an efficient algorithm for solving L gives an efficient algorithm for solving L_{easy}

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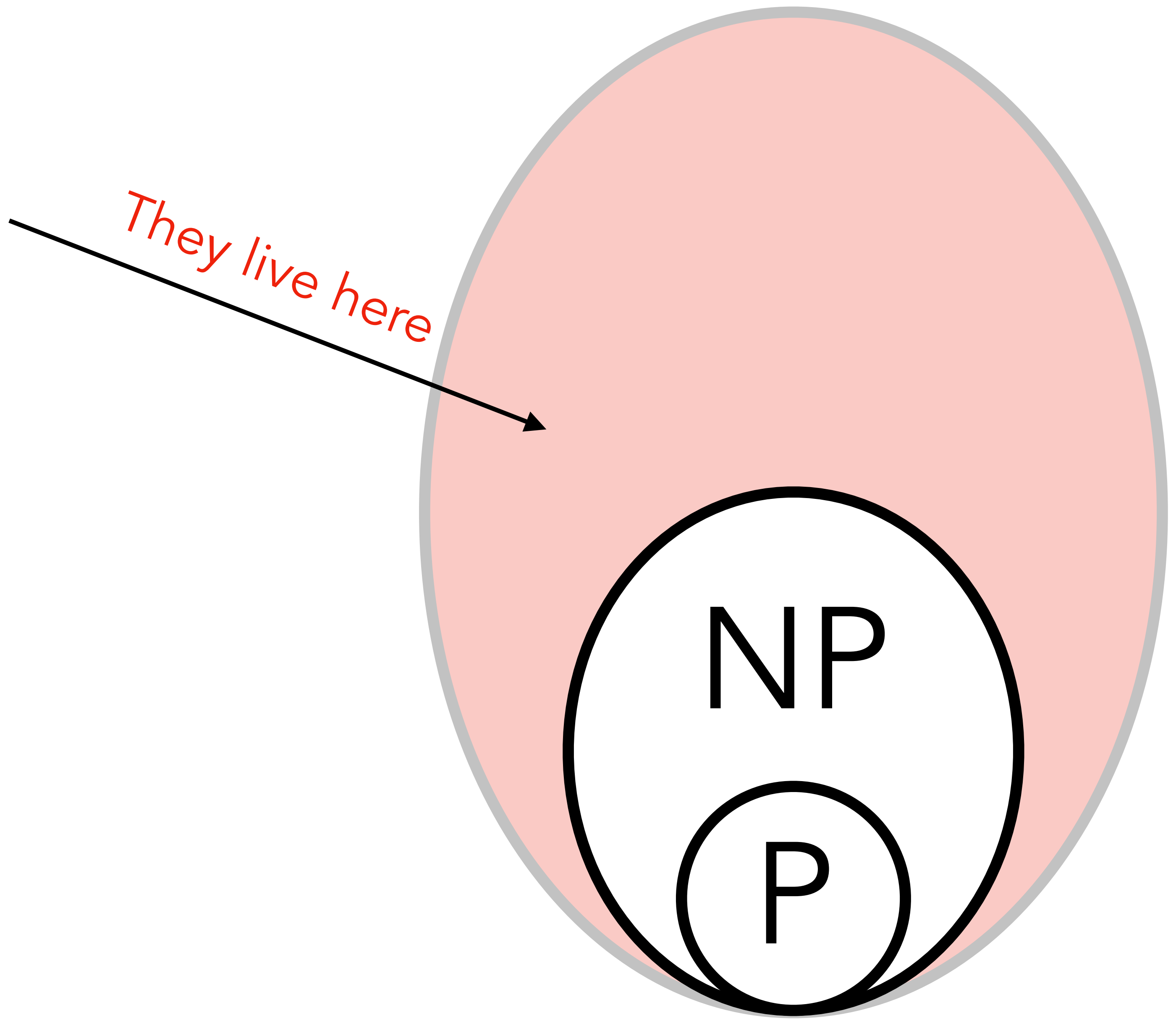
Such problems are called **NP-hard**



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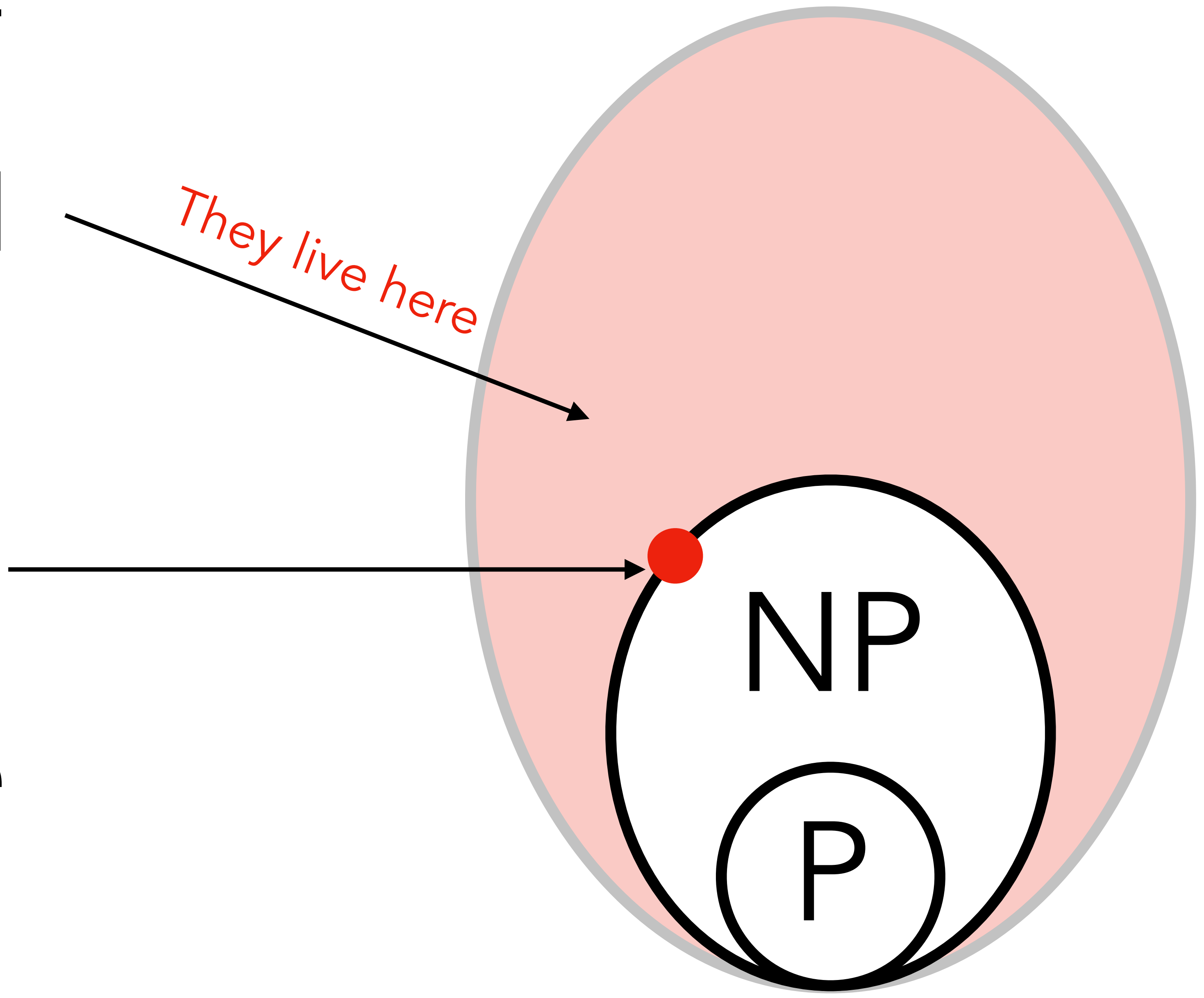

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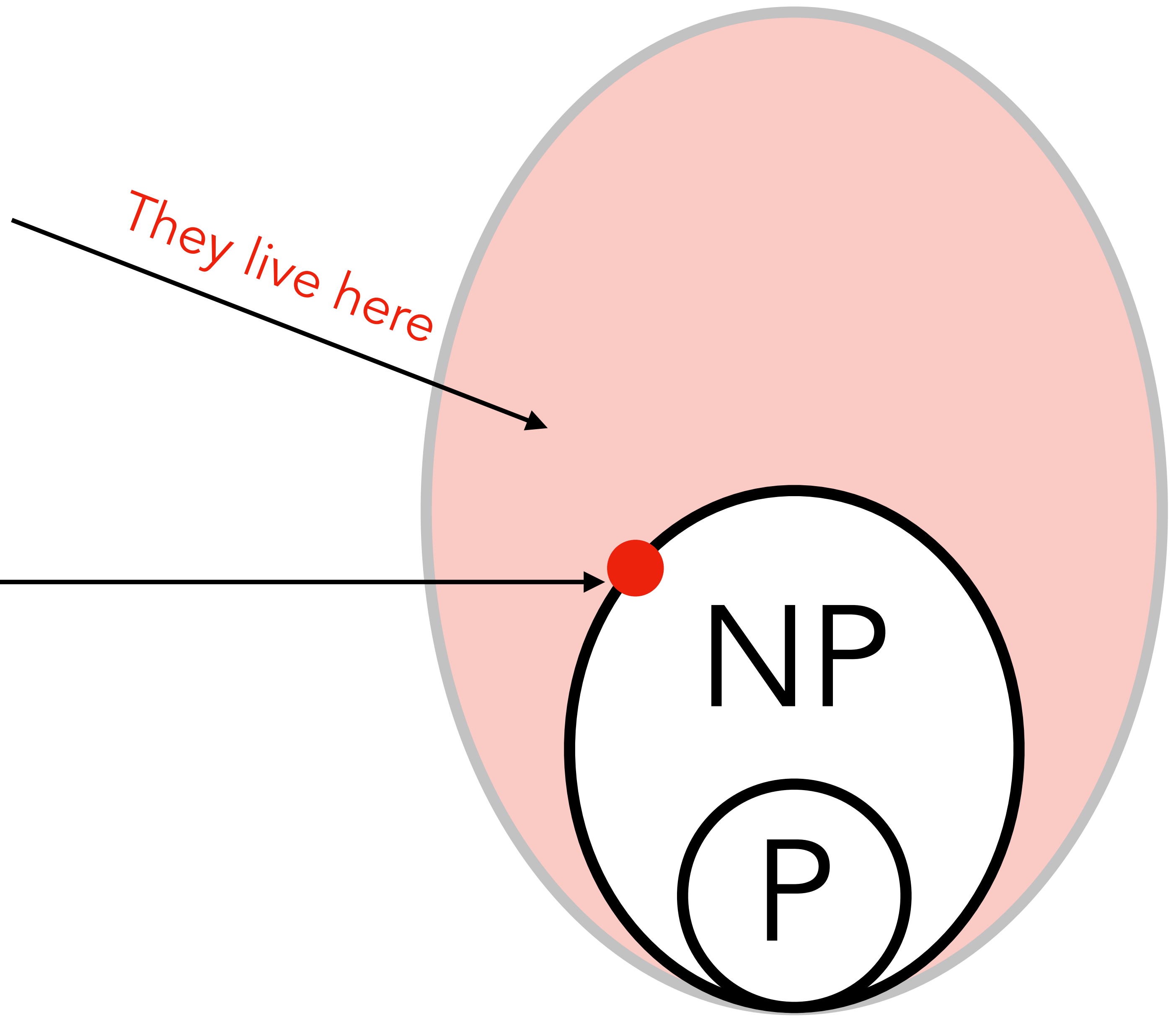
Some complexity classes are not known to have complete problems



More to come

- Oracles and Diagonalization
- Boolean Circuits
- Polynomial Hierarchy
- PCP Theorem
- Supplementary Homework: read Chapters 1 and 2