All problems are of equal value.

1. Recall that in lecture we showed that the monomial \( x_1 \cdots x_n \) requires a \( \sum \land_d \sum \) formula of size \( \geq \frac{2^n}{d+1} \). We explore here some improvements to this result.

   (a) Let \( f \) be homogeneous of degree \( d \) over \( \mathbb{F}[x] \), with a \( \land \sum \) expression \( f = \sum_i \alpha_i \ell_i(x)^{d_i} \) where \( \deg \ell_i \leq 1 \), and there is no bound on the \( d_i \). Prove that we can assume without loss of generality that \( d_i = d \), and that the \( \ell_i \) are homogeneous linear polynomials.

   \textit{Hint: Use the binomial theorem.}

   (b) Using (1a), prove that the monomial \( x_1 \cdots x_n \) requires a \( \land \sum \) formula with \( r \geq \frac{2^n}{n+1} \).

2. Give an explicit polynomial on \( n \) variables that requires \( 2^{\Omega(n)} \) size as a \( \land \sum \prod \) formula.

3. Prove that \( f = (\sum_i^n x_i y_i)^n \) has \( \land < \sum \leq \Omega(n) \).

4. Let \( f(\bar{x}, \bar{y}) \in \mathbb{F}[\bar{x}, \bar{y}] \) be a polynomial with the variable partition \( \bar{x}|\bar{y} \).

   (a) Prove that \( \text{coeff}_{\bar{y}|\bar{x}}(f) = \alpha_{\bar{y}} \cdot (\partial_{\bar{x}} f)_{\bar{x}=0} \), where \( \alpha_{\bar{y}} \) is a scalar that only depends on \( \bar{y} \), and prove that \( \alpha_{\bar{y}} \neq 0 \) in sufficiently large characteristic.

   That is, prove that by first differentiating \( f \) via \( \partial_{\bar{x}} \), and then setting all the variables in \( \bar{x} \) to zero, one can extract the corresponding coefficients in \( \bar{y} \).

   (b) Prove that \( \dim \text{coeff}_{\bar{y}|<\sum}(f) \leq \dim \land <\sum(f) \).