cs598maf Alg. and Geom. Complexity Theory	Out: *Mon., 2023-03-06*
Problem Set #3	
Prof. Michael A. Forbes	Due: *Wed., 2023-03-22 17:00*

All problems are of equal value.

1. Recall that in lecture we showed that the monomial $x_1 \cdots x_n$ requires a $\sum^r \bigwedge^d \sum$ formula of size $\geq \frac{2^n}{d+1}$. We explore here some improvements to this result.

   (a) Let f be *homogeneous* of degree d over $\mathbb{F}[\overline{x}]$, with a $\sum^r \bigwedge \sum$ expression $f = \sum_{i=1}^r \alpha_i \ell_i(\overline{x})^{d_i}$ where $\deg \ell_i \leq 1$, and there is *no bound* on the d_i. Prove that we can assume without loss of generality that $d_i = d$, and that the ℓ_i are homogeneous linear polynomials.

   Hint: Use the binomial theorem.

   (b) Using (1a), prove that the monomial $x_1 \cdots x_n$ requires a $\sum^r \bigwedge \sum$ formula with $r \geq \frac{2^n}{n+1}$.

   (c) Consider the following variant of the partial derivative measure, where we only consider derivatives of *exactly* a given order, $\dim \boldsymbol{\partial}^{=k}(f) = \dim\{\partial_{\overline{x}}^{\overline{a}} f\}_{\deg \overline{x}^{\overline{a}}=k}$.

      i. Prove an upper bound for $\dim \boldsymbol{\partial}^{=k}(\ell^d)$ for $\deg \ell \leq 1$.

      ii. Prove a lower bound for $\dim \boldsymbol{\partial}^{=k}(x_1 \cdots x_n)$.

      iii. By optimizing over k, prove that $x_1 \cdots x_n$ requires a $\sum^r \bigwedge \sum$ formula with $r \geq \Omega\left(\frac{2^n}{\sqrt{n}}\right)$.

      Hint: Use an estimate to binomial coefficients seen in lecture.

2. Give an explicit polynomial on n variables that requires $2^{\Omega(n)}$ size as a $\prod \sum \prod$ formula.

3. Prove that $f = (\sum_{i=1}^n x_i y_i)^n$ has $\dim \boldsymbol{\partial}^{<\infty}(f) \geq 2^{\Omega(n)}$.

4. Let $f(\overline{x}, \overline{y}) \in \mathbb{F}[\overline{x}, \overline{y}]$ be a polynomial with the variable partition $\overline{x}|\overline{y}$.

   (a) Prove that $\mathrm{coeff}_{\overline{y}|\overline{x}^{\overline{a}}}(f) = \alpha_{\overline{a}} \cdot (\partial_{\overline{x}}^{\overline{a}}(f))|_{\overline{x} \leftarrow \overline{0}}$, where $\alpha_{\overline{a}}$ is a scalar that only depends on $\overline{a}$, and prove that $\alpha_{\overline{a}} \neq 0$ in sufficiently large characteristic.

   That is, prove that by first differentiating f via $\partial_{\overline{x}}^{\overline{a}}$, and then setting all the variables in $\overline{x}$ to zero, one can extract the corresponding coefficients in $\overline{y}$.

   (b) Prove that $\dim \mathbf{coeff}_{\overline{y}|\overline{x}^{<\infty}}(f) \leq \dim \boldsymbol{\partial}^{<\infty}(f)$.