All problems are of equal value.

1. (BCS 21.6) For $n \in \mathbb{N}$ let $\text{SUM}_n := \sum_{i=1}^{n} x_i$ and $\text{PROD}_n := \prod_{i=1}^{n} x_i$. Show that the families $\text{SUM}$ and $\text{PROD}$ are not polynomial-size projections of each other.

2. (BCS 21.12) Define $f_n \in \mathbb{F}[x_1, \ldots, x_n]$ to be 0 when $n$ is not a power of 4, and otherwise inductively define $f_1 = x_1$, and

$$f_n = f_{n/4}(x_1, \ldots, x_{n/4}) \times f_{n/4}(x_{n/4+1}, \ldots, x_{n/2}) + f_{n/4}(x_{n/2+1}, \ldots, x_{3n/4}) \times f_{n/4}(x_{3n/4+1}, \ldots, x_n).$$

Thus $f_n$ is the polynomial computed by the complete binary tree with $n$ leaves and alternating layers of $\times$ and $+$. Show that the family $f = (f_n)_n$ is $\text{VF}$-complete, that is, $f \in \text{VF}$ and every $g \in \text{VF}$ is a polynomial-size projection of $f$.

3. Let $p$ be a prime. Suppose that for some $m$ and $k$, that the tensor rank of $m \times m$ matrix multiplication is at most $m^\tau$ over $\mathbb{F}_p^k$. Conclude that the exponent of matrix multiplication over the base field $\mathbb{F}_p$ is at most $\tau$.

   **Hint:** use tools developed in problem set 1.

4. Consider the bilinear map $\mu : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ defined by multiplying two univariate polynomials of degree $< n$ modulo $x^n$.

   (a) Write down an expression for the structural tensor of $\mu$.

   (b) Show that the rank of $\mu$ is at most $2n$ over any sufficiently large field.

   **Hint:** interpolation.

   (c) Show that the border rank of $\mu$ is at most $n$ over any sufficiently large field.