cs598maf Alg. and Geom. Complexity Theory	Out: *Thu., 2023-02-16*
Problem Set #2	
Prof. Michael A. Forbes	Due: *Wed., 2023-03-01 17:00*

All problems are of equal value.

1. (BCS 21.6) For $n \in \mathbb{N}$ let $\mathrm{SUM}_n := \sum_{i=1}^n x_i$ and $\mathrm{PROD}_n := \prod_{i=1}^n x_i$. Show that the families SUM and PROD are not polynomial-size projections of each other.

2. (BCS 21.12) Define $f_n \in \mathbb{F}[x_1, \ldots, x_n]$ to be 0 when n is not a power of 4, and otherwise inductively define $f_1 = x_1$, and

$$f_n = f_{n/4}(x_1 \ldots, x_{n/4}) \times f_{n/4}(x_{n/4+1}, \ldots, x_{n/2}) \\ + f_{n/4}(x_{n/2+1}, \ldots, x_{3n/4}) \times f_{n/4}(x_{3n/4+1}, \ldots, x_n) \ .$$

   Thus f_n is the polynomial computed by the complete binary tree with n leaves and alternating layers of $\times$ and $+$. Show that the family $f = (f_n)_n$ is VF-complete, that is, $f \in \mathsf{VF}$ and every $g \in \mathsf{VF}$ is a polynomial-size projection of f.

3. Let p be a prime. Suppose that for some m and k, that the tensor rank of $m \times m$ matrix multiplication is at most m^τ over $\mathbb{F}_{p^k}$. Conclude that the exponent of matrix multiplication over the base field $\mathbb{F}_p$ is at most τ.

   Hint: use tools developed in problem set 1.

4. Consider the bilinear map $\mu : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ defined by multiplying two univariate polynomials of degree $< n$ modulo x^n.

   (a) Write down an expression for the structural tensor of μ.

   (b) Show that the rank of μ is at most $2n$ over any sufficiently large field.
   Hint: interpolation.

   (c) Show that the border rank of μ is at most n over any sufficiently large field.