depth reduction vs fan-in

Q: how to measure only circuit complexity? [fan-in]

Q: size vs depth? [traded]

Q (depth reduction): for small fan-in, is small depth? size is similar for bounded fan-in.

prop: if has fan-in 2 formula, size 5 ⇒ has fan-in 2 formula depth O(\(2^n\)).

pf: if rooted, fan-in \(2^n\) depth \(d\). ⇒ has size \(O(2^d)\) formula of fan-in 2

\[
\begin{array}{c}
\text{in rooted} \\
\text{fan-in } 2^n \\
\text{depth } d \\
\end{array}
\]\

⇒ \(2^d\) nodes in depth \(d\) \(\leq 2^n\) \(\Rightarrow d \leq \log_2(2^n) = n\).

pf: in rooted formula, size \(s = \frac{1}{2} \times \text{fan-in}^2 \), depth \(d \leq \log_2(\frac{s}{2})\).

prop: if has fan-in 2 formula, size \(s = 1 + \text{fan-in}^2 \), depth \(d \leq \log_2(\frac{s}{2})\). in rooted, size \(s = \frac{1}{2} \times \text{fan-in}^2 \), depth \(d \leq \log_2(\frac{s}{2})\).

pf: idea: recursive re-balancing.

Q: fan-in of depth 2 form, size \(s = \frac{1}{2} \times \text{fan-in}^2 \), depth \(d \leq \log_2(\frac{s}{2})\).

pf: find balanced part?

lem: formula \(f\) fan-in 2 \(\Rightarrow\) leaf, exists node \(u \not\in f\) has \(\frac{1}{2}L(u) < L\).

pf: if \(u \not\in f\), \(L(u) < L(f)\) \(\Rightarrow\) leaf.

\[
\begin{array}{c}
\text{while} \\
u \in v \ast w \\
\text{if } L(u) \leq L(v) \text{, output } v \\
\text{else } u \leftarrow \text{arg max } L \\
\end{array}
\]
complexity: \( \Theta \).

\[ f(x, y) = \Delta_x \]

where \( g, h \) have formula \( \Delta \) and \( \Delta \) is poly in \( x, y \).

\[ f = f_{\text{poly}} \]

\[ h = f_{\text{poly}} \]

\[ p = f_{\text{poly}} - f_{\text{poly}} = g - h \]

\[ \text{w/ } v \text{ & } w \text{ on } \Delta \]

\[ f = O(1) \text{ depth} \]

f fan-in 2 formula \( \Delta \) leaves \( v, w \): \( \frac{1}{2} L \leq L \leq \frac{2}{3} L \)

\[ \Rightarrow f_{\Delta} \text{ has } v \leq L - L_{\Delta} + 1 \leq \frac{2}{3} L + 2 \]

\[ = \frac{2}{3} L + 2 \]

\[ \Rightarrow f_{\Delta} \text{ is a proper subformula} \]

\[ \Rightarrow \text{depth}(L) \leq O(1) + \text{depth}(\frac{2}{3} L + 2) \leq \cdots \leq O(L) \] \[ \Rightarrow \text{small formula is } O(n) \text{-depth?} \]

\[ \Rightarrow \text{an electron machine program (EMP) is a labelled directed acyclic graph with} \]

\[ \text{node } \text{ : source } 1 \text{ form } 2 \]

\[ \text{sink } \text{ : form } 1 \text{ and the node \#1} \]

\[ \text{edges } \text{ : labelled } / \text{ } \alpha \leq \overline{\alpha} \]

The size is \#node. A node \( i \) consumes \( f_i = \frac{1}{i!} \text{ in } \text{time} \)

\[ g_i = \frac{x^i - y^i}{i!} \]

\[ \text{let } f \text{ has fan-in size } s \Rightarrow f \text{ has } \text{EMP } \text{ of } s + 1 \text{ nodes} \]

\[ f \text{ : } \text{closed recursion on } \alpha \text{ family} \]

\[ \text{before } \text{ : source } 1 \text{ form } 2 \]

\[ \text{after } \text{ : source } 1 \text{ form } 2 \]

\[ f_2 = g \cdot h \]

\[ f_i = f_{\text{poly}} \]

\[ f_{\text{poly}} \text{ is linear in } y \]

\[ p \text{ & } h \]
Def: A layered homogeneous ABP as

\[ \forall x \in \mathbb{R}^n, \quad v(x) = \sum_{v \in p} w_{v} \cdot \phi_{v}(x) \]

all edges labelled by linear maps \( E : \xi \times \xi \to V_j \) s.t. \( v_j \subseteq V_i \), \( v_i \subseteq V_j \)

let: all nodes \( v_i \) in layered homogeneous ABP. \( \forall x \in \mathbb{R}^n \)

claim:

\[ \sum_{v_i \in p} w_{v_i} \cdot \phi_{v_i}(x) \]

prop: if degree \( d \), ABP \( v_i \in \mathbb{R}^n \) \( \forall x \in \mathbb{R}^n \).

clf: idea: geometric simulation

clm: \( v_j \) computes \( H_j(v_j) \) if build ABP

\[ \sum_{v_i \in p} w_{v_i} \cdot \phi_{v_i}(x) \]

if correctly?

\[ \sum_{v_i \in p} w_{v_i} \cdot \phi_{v_i}(x) \]

linear segments

\[ \sum_{v_i \in p} w_{v_i} \cdot \phi_{v_i}(x) \]

form

\[ \sum_{v_i \in p} w_{v_i} \cdot \phi_{v_i}(x) \]

form
\[ h_0(v) + E_\mu \left( \sum_{v_i \in \text{label}(v)} \sum_{\forall \mu \in \text{label}(v)} \right) \]

H_{v+1}(\text{value}(v)) = H_v(\text{value}(v) \cup \text{label}(u_j \rightarrow v_{j+1}), \text{label}(u_j \rightarrow v_{j+1}))

\[
= \sum_{v_i \in \text{label}(v)} H_v(\text{value}(u_i) \cup \text{label}(u_j \rightarrow v_{j+1}))
\]

= value(u_j) \cup \text{label}(u_j \rightarrow v_{j+1})

\]

Explanation: s-(ken) nodes

Proof: f : d \rightarrow \text{layered hang AEP size } s = f^2 = \left( \prod_{i=1}^{d} M_i(x) \right)_{s \times s}

\[
\text{since } M_i \text{ is } s \times s \text{ only matrix of } V_i \times V_{i+1} = \text{encapsulate linear layer}
\]

Sketch: standard connection between intiated matrix multiplication and paths in graphs

Prop: f : d \rightarrow \text{layered hang AEP size } s \rightarrow f \text{ has depth } d \text{ formula size } O(s^2 d)

pf: f = \left( \prod_{i=1}^{d} M_i(x) \right)_{s \times s} = \left( \left( \prod_{i=1}^{d} M_i(x) \right)_{s \times s} \right)_{s \times s}

\[
\text{for } M_i \text{ many terms }\]

\[
\text{sum over all paths }\]

- \left( \text{sum } \left( M_i \right) \right)_{s \times s}

\text{A formula } = \text{AEP}

Prop: f : d \rightarrow \text{AEP size } s \rightarrow f \text{ has } O(\text{lg } d) \text{ depth of bounded linear size } 5(s)(\text{sel})

Sketch: multiply matrix rows (new) rows of AEP

\[
= O(\text{lg } d) \text{ depth } [\text{formula size } s]
\]

therefore: \text{bounded, carefully check}

\text{use close diagonalization}

\text{L property, formula size } 2^{O(\text{lg } d)}

\text{pf:2 operations}

\text{similar result in } \text{depths } \text{unlimited to degree of phys, char?}

\text{its another day, we know, I will see interest}