10 min over w/ skips at end

$5 \to 7$

depth reduction vs det

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CS595 maf Algebraic and Geometric Complexity Theory : lecture 4 (2023-01-31)

logistics : part 1 due 02-06

last lecture : constructions of alg ckts [trivial] [esym dyn prog] [Ryser]
- det [laplace expansion] [esym interpolation]
- perm [gaussian elim] [↑ w/o names]
- esym

today: depth reduction

Q : how to measure alg ckt complexity? [size] [depth] [tradeoffs?]

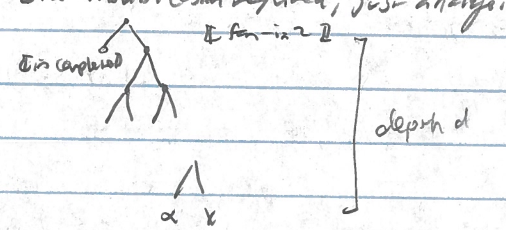
Q - size vs depth?

Q (depth reduction) : f small ckt $\overset{?}{\Rightarrow}$ f small ckt w/ small depth? [will do for $\boxed{\text{formulas}}$] [ckt results similar, but longer] [limits?]

lem : f has $\boxed{\text{formula}}$ fan-in $\boxed{2}$ depth d. $\Rightarrow$ f has size $O(2^d)$ formula w/ fan-in 2

[no modification required, just analysis]

pf.



$\Rightarrow$ # nodes on depth $i \le 2^i$

nodes $\le \sum_{i=0}^{d} 2^i \le O(2^d)$

□

prop. f has fan-in 2 formula, size s $\Rightarrow$ f has fan-in 2 formula depth $O(\lg s)$ $\Downarrow$ has size poly(s) [acceptable]

rmk : - depth $\Omega(\lg s)$ required w/o improving ckt size [need better measure]

lem : f has fan-in 2 formula w/ size s, L leaves. then $s = \Theta(L)$ [allows us to use L instead]

[statement about trees]

pf : $L \le S$: each leaf is node

$s \le 2L - 1$: wlog no fan-in 1 nodes

by tree recursion

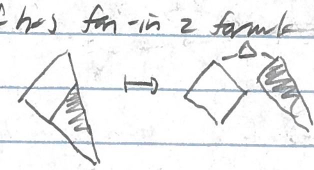
f leaf : $s = 1$, $L = 1$, $s = 2L - 1$

$f = g * h$: $s(f) = 1 + \underbrace{s(g)}_{\le 2L(g)-1} + \underbrace{s(h)}_{\le 2L(h)-1} \le 2(L(g) + L(h)) - 1$ [induction] $= L(F)$ [star] [op] [formula] eq when? .

- Trembaczk

prop : f has fan-in 2 formula, size s $\Rightarrow$ f has fan-in 2 formula depth $O(\lg s)$

pf. idea : recursive rebalancing



$\swarrow$ recurse fig.

[not quite this clean]

Q : find $\boxed{\text{balanced}}$ pivot?

lem formula f fan-in 2 w/ L leaves. exists node v w/ f_v has $\frac{1}{3}L \le L_v < \frac{2}{3}L$ [sub formula rooted at v] [$L(f_v)$]

[really about trees]

pf : algo - $v \leftarrow$ root of formula

while

$v \leftarrow v * w$

if $L_v < 2/3 L$, output v

else $v \leftarrow \arg\max_{t \in \{v, w\}} L_t$

correctness : $L_v \ge 1/3 \cdot L$ [invariant]

$L_v \ge 1/3 L$ $\Rightarrow$ v not leaf [$L \gg 0$]

$L_v \ge 2/3 L \Rightarrow$ correct

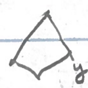
$L_v \ge 2/3 L$, $L_v = L_v + L_w$ $\Rightarrow L_t \ge 1/3 L$ [restore invariant]

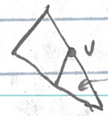
decreasing

complexity: irrelevance

lem: f formula fan-in 2, v node in f. Then $f = (g-h)\cdot f_v + h$

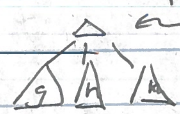
where g, h have formulas w/ $\leq L(f|_{v=y})$ leaves ← replace v w/ new var

pf: $f|_{v=y}$ =  is poly in $\bar{x}, y$

 f_v ← y appears $\boxed{\text{once}}$ $\Longrightarrow$ (← f formula) $f|_{v=y}$ is linear in y = $p\,y + h$

$h = f|_{v=0}$

$p = f|_{v=1} - f|_{v=0} = g - h$ ← interpolation; #leaves $\leq L(f|_{v=y})$ □

f =  ← $O(1)$ depth

f fan-in 2 formula L leaves w/ pivot v, $\frac{1}{3}L \leq L_v < \frac{2}{3}L$

$\Rightarrow f|_{v=y}$ has $\leq \underbrace{L - L_v}_{\text{remove subformula}} + \underbrace{1}_{\text{keep } v=y} \leq 2/3\, L + 1$

$\Rightarrow g, h$ ——

$\Rightarrow \text{depth}(L) \leq O(1) + \text{depth}(2/3\, L + 1) \leq \cdots \leq O(\lg L)$ ∎

Q: small formulas vs $O(1)$-depth? [introduce new model to help]

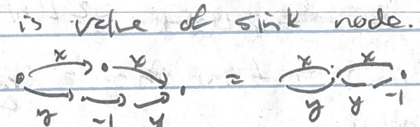
def: an algebraic branching program (ABP) is a labelled directed acyclic graph with

nodes: source s w/ fan-in zero
sink t w/ fan-out zero [no other node labels]

edges: labelled w/ $\alpha \in \mathbb{F}$
x_i, $i \in [n]$

The size is #nodes. A node v computes $\sum_{p: s \rightsquigarrow v} \prod_{e \in p} \text{label}(e)$ [path]

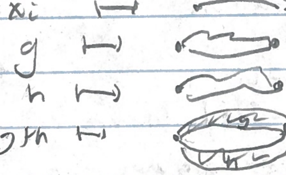
the output is value of sink node.

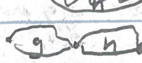
eg: $x^2 - y^2$ = 

lem f has formula size $s \Longrightarrow f$ has ABP w/ $s+1$ nodes

pf. idea: recursion on formulas

leaves: size = 1
$\alpha \mapsto \bullet \xrightarrow{\alpha} \bullet$ 2 nodes
$x_i \mapsto \xrightarrow{x_i}$ 2 nodes

$f = g + h$: 
g: $s(g)+1$
h: $s(h)+1$
$g+h$ [parallel] $s(g) + s(h) + 2 - 2 = s(f) + 1 \leq s(f)+1$

$f = g \cdot h$: $g \cdot h \mapsto$  [series] $s(g) + s(h) + 2 - 1 = s(f) \leq$ ——

[structural results]

def: a <u>layered homogeneous ABP</u> is ______

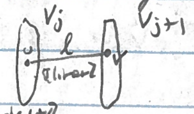
vertices form partition $V_0 \sqcup V_1 \sqcup \cdots \sqcup V_d$, $s \in V_0$, $t \in V_d$

zero edge labels

all edges labelled w/ linear forms $\sum_i \alpha_i x_i$, go from V_j to V_{j+1} some j

lem: all node values in layered homog ABP are homog ~~$O(n)$ complexity~~

sketch:



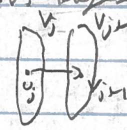
[induct] deg $j \to$ deg $j+1$

[main]

[$k=0$ is constant, technical edge case]

prop: f degree d, ABP w/ s nodes $\Rightarrow$ any $k \geq 1$ $H_k(f)$ has layered homog ABP w/ $O(s \cdot k)$ nodes

pf: idea: gate simulation



$v \mapsto v_0, \ldots, v_k$

clm: v_j computes $H_j(v)$ [build ABP as we prove]

pf: idea:

$u \xrightarrow{x_i} v \longmapsto$ [deg j] $u_j \xrightarrow{x_i} v_{j+1}$ [deg $j+1$]

$w \xrightarrow{\alpha} v \longmapsto$? [no constants on edges]

$\uparrow x_{i'}$

u

[shortcut] $\longmapsto$ $u_j \xrightarrow{\alpha x_{i'}} v_{j+1}$

[merge] $\longmapsto$ $u_j \xrightarrow{x_i + \alpha x_{i'}} v_{j+1}$

[do systematically?]

<u>idea</u>: $p: s \rightsquigarrow t$ looks like

$$s \rightsquigarrow s^{(1)} \xrightarrow{x_{i_1}} t^{(1)} \rightsquigarrow s^{(2)} \xrightarrow{x_{i_2}} t^{(2)} \rightsquigarrow \cdots \rightsquigarrow s^{(d)} \xrightarrow{x_{i_d}} t^{(d)} \rightsquigarrow t$$

w/ "$\rightsquigarrow$" only using $\boxed{\text{constants}}$ [we want to shortcut these paths]

$\Rightarrow$ (unique) decomposition – initial segment $s \rightsquigarrow s^{(1)} \xrightarrow{x_{i_1}} t^{(1)} \rightsquigarrow s^{(2)}$ [sandwich]

– later segments $i \geq 2$ $s^{(i)} \xrightarrow{x_i} t^{(i)} \rightsquigarrow s^{(i+1)}$

w/ $s^{(d+1)} = t$ [open face]

<u>linear terms</u>: [initial segments]

label $s_0 \longrightarrow v_1$ [source] with [linear part of v]

$$\sum_{\substack{p: s \overset{\mathbb{F}}{\rightsquigarrow} w \\ w \xrightarrow{x_i} r \\ p': r \overset{\mathbb{F}}{\rightsquigarrow} v}} \left(\prod_{e \in p} \text{label}(e)\right) \cdot x_i \cdot \left(\prod_{e \in p'} \text{label}(e)\right)$$

$\Rightarrow$ value$(v_1) = H_1(v)$ [clear] $\underbrace{}_{\text{linear form}}$

<u>higher degree parts</u>: [later segments] $j \geq 1$

label $u_j \longrightarrow v_{j+1}$ w/ $\displaystyle\sum_{\substack{w \xrightarrow{x_i} r \\ p: r \overset{\mathbb{F}}{\rightsquigarrow} v}} x_i \cdot \left(\prod_{e \in p} \text{label}(e)\right)$ [one-sided]

[correctness]

$$\text{value}(v) = \sum_{p: s \rightsquigarrow v} \prod_{e \in p} \text{label}(e)$$

$$= H_0(v) + \sum_{\substack{p: s \rightsquigarrow u \\ u \xrightarrow{x_i} w \\ p': w \overset{\mathbb{F}}{\rightsquigarrow} v}} \prod_{e \in p} \text{label}(e) \cdot x_i \cdot \prod_{e \in p'} \text{label}(e)$$

[constant term] [last nonconstant] [unique decomp]

today: depth reduction
- fan-in 2 formulas
- formulas vs depth 4 [via ABPs]

logistics: pset-2 due 02-08

next lecture: complexity

$$= H_0(v) + \sum_u \underbrace{\Big(\sum_{p: s \rightsquigarrow u} \prod_{e \in p} \mathrm{label}(e)\Big)}_{\mathrm{value}(u)} \cdot \underbrace{\Big(\sum_{\substack{u \rightsquigarrow w \\ p: w \rightsquigarrow v}} x_i \cdot \prod_{e \in p} \mathrm{label}(e)\Big)}_{\mathrm{label}\ u_j \to v_{j+1}\ [\text{any } j]}$$

$$H_{j+1}(\mathrm{value}(v)) = H_{j+1}(\underline{\qquad\qquad})$$

$$\overset{[\text{constant dies}]}{=} H_j\Big(\sum_u \mathrm{value}(u) \cdot \underbrace{\mathrm{label}(u_j \to v_{j+1})}_{\text{homog deg} = 1}\Big)$$

$$\overset{[\text{take homog parts}]}{=} \sum_u \underbrace{H_j(\mathrm{value}(u))}_{= \mathrm{value}(u_j)\ [\text{induction}]} \cdot \mathrm{label}(u_j \to v_{j+1})$$

$$= \mathrm{value}(v_{j+1}) \quad [\text{def of ABP value}]$$

correctness: output value of $H_k(t)$ via new sink t_k
vertices are layered
edge labels linear forms

complexity: $s \cdot (k+1)$ nodes ☐

prop: f deg d w/ layered homog ABP size $s \Rightarrow f = \left(\prod_{i=1}^{d} M_i(\bar{x})\right)_{1,s}$

where M_i is $s \times s$ adj matrix of $V_i \times V_{i+1} \Rightarrow$ entries are linear forms
source is vertex "1"
sink is vertex "s"

sketch [exc 4]: standard connection between iterated matrix multiplication and paths in graphs

prop: f deg d w/ layered homog ABP size $s \Rightarrow f$ has depth 4 formula size $s^{O(\sqrt{d})}$

pf: $f = \left(\prod_{i=1}^{d} M_i(\bar{x})\right)_{1,s} = \big(([\,][\,] \cdots [\,]) \cdots (\quad [\,]) \big)_{1,s}$

$$= \Big(\prod_{i=1}^{\sqrt{d}} \underbrace{\prod_{j=1}^{\sqrt{d}} M_{\sqrt{d} \cdot i + j}(\bar{x})}\Big)_{1,s}$$

sum over $s^{\sqrt{d}}$ paths $\Rightarrow$ depth-2 formula of size $\mathrm{poly}(s^{\sqrt{d}}, n)$
each edge
$s^2 \cdot \sqrt{d}$ many formulas
sum over $s^{\sqrt{d}}$ paths $\Rightarrow$ ______

=  $\Leftarrow$ size $\mathrm{poly}(s^{\sqrt{d}}, n)$ ☐

cor: f deg d formula size $s \Rightarrow$ ______ [formula $\Rightarrow$ ABP $\Rightarrow$ layered homog ABP $\Rightarrow$ depth 4]

prop: f deg d ABP size $s \Rightarrow f$ has $O(\lg s d)$ depth unbounded fan-in formula size $s^{O(\lg s d)}$
______ ckt size $\mathrm{poly}(s)$

sketch: multiply matrices [two] blocks at a time
$\Rightarrow O(\lg s d)$ depth [$\Rightarrow$ formula size]
[memoize] for small ckt

thm: ____ ckt ____ $\Rightarrow f$ has $O(\lg s d)$ depth unbounded fan-in formula size $s^{O(\lg s d)}$
______ ckt size $\mathrm{poly}(s)$
f has depth 4 ______ formula size $s^{O(\sqrt{d})}$

sketch: balance ckt, carefully ____ 3 ____
use clever identities

cor: perm requires $2^{\Omega(n)}$ depth 4 formulas $\Rightarrow$ perm requires $2^{\Omega(\sqrt{n})}$ [general] ckts [if] $\mathrm{char}(\mathbb{F}) = 0$
____ $2^{\omega(\sqrt{n} \lg n)}$ ____ $n^{\omega(1)}$ ____
[positive] [negative]

rmk: - similar results in [boolean] ckts unlikely [low degree of polys, ckt]
- lbs [almost] day are known [will see ideas]